Structured derivations

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Background

- Structured derivations combine Dijkstra's calculational style proofs with forward and backward proofs
  - intended as a high-level intuitive format for presenting mathematical arguments
  - main target: students in junior high school, high school and university introductory courses
  - level of formality in proofs can be freely chosen (i.e., allows also non-axiomatic proofs)
- Structured derivations have an exact syntax definition
  - gives guidance to students on how to formulate mathematical arguments
  - makes it easier to follow a mathematical argument, and to find errors in own argument
  - syntax works in all languages (by using special symbols)
  - can also have verbose syntax, e.g. to explain meaning of constructs
- Structured derivations proposed initially by Back and von Wright in 1998

Checking correctness of structured derivation

- A structured derivation can be reduced to a natural deduction proof
- Structured derivations allow computer based checking of derivations
    - Interactive and automatic proof checkers: PVS, Isabelle,
    - SMT solvers: Yices and Z3,
    - CAS: Mathematica, Sage
- In practice, requires syntax analysis to transform standard mathematical notation to formulas that are understandable by mechanized proof systems
    - and taking into account different national traditions for mathematical notation

Example from analytic geometry

The problem is to determine the point on the parabola $y = x^2 - 2x - 3$ where the tangent of the parabola has a 45° slope.

We reformulate the problem as follows:

> *Determine the coordinates* (x, y) *on the parabola* f*, defined by* $f(x) = x^2 - 2x - 3$ *for* x *in* $\mathbb{R}$ *, where the slope* α *of the tangent of the parabola is* 45°*.*

We use observations to establish some preliminary results, before we solve the main problem.

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[2] $\{$determine $x\}$

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$\equiv$ $\{$assumption (b), derivative of f is defined by $f'(x) = 2x - 2$, for $x \in \mathbb{R}\}$

$2x - 2 = 1$

[2] {determine x}

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	$2x - 2 = 1$
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	$x = \frac{3}{2}$

... $x = \frac{3}{2}$

$\Vdash \qquad (x, y)$

$$\begin{array}{ll} \Vdash & (x, y) \\ = & \{\text{observation [2]}\} \end{array}$$

$$
\begin{aligned}
\Vdash \quad & (x, y) \\
= \quad & \{\text{observation [2]}\} \\
& (\tfrac{3}{2}, y)
\end{aligned}
$$

$$
\begin{array}{ll}
\Vdash & (x, y) \\
= & \{\text{observation [2]}\} \\
 & (\tfrac{3}{2}, y) \\
= & \{\text{assumption (a) and observation [2]}\}
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 & (\frac{3}{2}, (\frac{3}{2})^2 - 2(\frac{3}{2}) - 3)
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$\square$

$\Vdash \quad (x, y)$

$= \quad$ {observation [2]}

$\quad (\frac{3}{2}, y)$

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$\quad (\frac{3}{2}, (\frac{3}{2})^2 - 2(\frac{3}{2}) - 3)$

$= \quad$ {simplify}

$\quad (\frac{3}{2}, -\frac{15}{4})$

$\square$

- The point we are looking for is thus $(x, y) = (\frac{3}{2}, -\frac{15}{4})$.

Derivation structure

•	Determine the point (x, y), where	*task*
(a)	parabola f is defined by $f(x) = x^2 - 2x - 3$, for $x \in \mathbb{R}$, and	*assumptions*
(b)	(x, y) is on the parabola f, and	
(c)	tangent of the parabola in point (x, y) has slope $\alpha = 45^\circ$	
[1]	{determine the first derivative of f in point x}	*observation*

calculation as subderivation

•	assumption (c)
$\equiv$	{inclination coefficient $k = \tan\alpha$}
	tangent of the parabola has slope coefficient $\tan 45^\circ$
$\equiv$	$\{\tan 45^\circ = 1\}$
	tangent of the parabola has slope coefficient 1 in point (x, y)
$\equiv$	{first derivative gives the slope coefficient}
	$f'(x) = 1$

... $f'(x) = 1$

[2] {determine x} *observation*

calculation as subderivation

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\begin{array}{ll}
\bullet & f'(x) = 1 \\
\equiv & \{\text{assumption (a), derivative of } f \text{ is defined by } f'(x) = 2x - 2 \text{, for } x \in \mathbb{R}\} \\
& 2x - 2 = 1 \\
\equiv & \{\text{solve } x\} \\
& x = \dfrac{3}{2}
\end{array}
$$

$\ldots$ $x = \dfrac{3}{2}$

$\Vdash$ *solving task with calculation*

$$
\begin{array}{ll}
& (x, y) \\
= & \{\text{observation [2]}\} \\
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= & \{\text{simplify}\} \\
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\end{array}
$$

$\square$ *End of derivation*

Syntax of structured derivations

- Two main syntactic categories
  - derivations
  - justifications
- Allows building arbitrary deep and arbitrary long derivations

derivation:

$\bullet$	*question*
-	*assumption*
$\vdots$	
-	*assumption*
$+$	*justification*
	proposition
$\vdots$	
$+$	*justification*
	proposition
$\Vdash$	*justification*
	term
rel	*justification*
	term
$\vdots$	
rel	*justification*
	term
$\square$	*answer*

justification:

{reason}

derivation

$\vdots$

derivation

Combining proof paradigms

- Structured derivations combine three main proof paradigms
  - forward proofs (observations)
  - backward proofs (justifications with subderivations)
  - calculations
- Different proof paradigms can be used in same derivation
  - proof steps are different,
  - some arguments are best done as calculations
  - sometimes forward proofs are better, to gather facts needed later on,
  - sometimes backward steps are needed, as proof strategies for breaking up the proof in smaller, more manageable parts

derivation:

$\bullet$ *question*
\- *assumption*

$\vdots$

\- *assumption*
$+$ *justification* *(forward)*
proposition

$\vdots$

$+$ *justification*
proposition
$\Vdash$ *justification* *(backward)*
term *(calculation)*
rel *justification*
term

$\vdots$

rel *justification*
term
$\Box$ *answer*

justification:

{reason}
derivation *(subderivation)*

$\vdots$

derivation *(subderivation)*

Logic in high school

- Logic is everywhere in the high school curriculum
  - But it is hidden in informal arguments
  - This makes it more difficult for students to understands the logical arguments, they have to re-invent the logic for themselves
  - The best students will get it, many will never do that
  - In stead, they learn the argument templates by hearth, and apply them without understanding
- Propositional connectives and quantifiers can be used to structure the mathematical argument
  - example: conjunction and disjunction separate the argument into two independent parts
  - implication is proved by making additional assumptions
  - natural deduction inference rules are good proof strategies for partitioning the proof into smaller parts

Example of logic in high school

We show an example of how logic can be used in practice in high school math.

The problem is to solve the inequality $(1+x)^2 \leq 1$

- $(1+x)^2 \leq 1$

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\equiv & \{\text{zero product rule: } (ab = 0) \equiv (a = 0 \vee b = 0)\} \\
& x = 0 \vee 2 + x = 0 \\
\equiv & \{\text{subtract 2 from both sides in the right conjunct}\}
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\equiv & \{\text{rewrite the inequalities in an alternative form: } (a \leq 0) \equiv (a = 0 \vee a < 0) \\
 & x(2+x) = 0 \vee x(2+x) < 0 \\
\equiv & \{\text{solve the two disjuncts separately}\} \\
 & \begin{array}{ll}
 \bullet & x(2+x) = 0 \\
 \equiv & \{\text{zero product rule: } (ab = 0) \equiv (a = 0 \vee b = 0)\} \\
  & x = 0 \vee 2 + x = 0 \\
 \equiv & \{\text{subtract 2 from both sides in the right conjunct}\} \\
  & x = 0 \vee x = -2
 \end{array}
\end{array}
$$

- $x(2+x) < 0$

$\bullet$ $\quad x(2+x) < 0$

$\equiv$ $\quad$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \land b > 0) \lor (a > 0 \land b < 0)$}

$\bullet \quad x(2+x) < 0$

$\equiv \quad$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$(x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\bullet$ $\quad x(2 + x) < 0$

$\equiv$ $\quad$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$\quad (x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\equiv$ $\quad$ {simplify both disjuncts}

$\bullet$	$x(2+x) < 0$
$\equiv$	{a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}
	$(x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$
$\equiv$	{simplify both disjuncts}
	$(x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$

$\bullet \quad x(2+x) < 0$

$\equiv \quad$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$\quad (x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\equiv \quad$ {simplify both disjuncts}

$\quad (x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$

$\equiv \quad$ {the right disjunct is false for each x}

- $x(2+x) < 0$

$\equiv$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$(x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\equiv$ {simplify both disjuncts}

$(x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$

$\equiv$ {the right disjunct is false for each x}

$(-2 < x < 0) \vee F$

- $x(2+x) < 0$

$\equiv$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$(x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\equiv$ {simplify both disjuncts}

$(x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$

$\equiv$ {the right disjunct is false for each x}

$(-2 < x < 0) \vee F$

$\equiv$ $\{p \vee F \equiv p\}$

$\bullet$	$x(2+x) < 0$
$\equiv$	{a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}
	$(x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$
$\equiv$	{simplify both disjuncts}
	$(x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$
$\equiv$	{the right disjunct is false for each x}
	$(-2 < x < 0) \vee F$
$\equiv$	$\{p \vee F \equiv p\}$
	$-2 < x < 0$

$$
\begin{array}{ll}
\bullet & x(2+x) < 0 \\
\equiv & \{\text{a product is negative iff one factor is positive and the other is} \\
& \text{negative: } (ab<0) \equiv (a<0 \wedge b>0) \vee (a>0 \wedge b<0)\} \\
& (x<0 \wedge 2+x>0) \vee (x>0 \wedge 2+x<0) \\
\equiv & \{\text{simplify both disjuncts}\} \\
& (x<0 \wedge x>-2) \vee (x>0 \wedge x<-2) \\
\equiv & \{\text{the right disjunct is false for each } x\} \\
& (-2<x<0) \vee F \\
\equiv & \{p \vee F \equiv p\} \\
& -2<x<0
\end{array}
$$

$$
\ldots \quad (x=0 \vee x=-2) \vee (-2<x<0)
$$

$\bullet \quad x(2+x) < 0$

$\equiv \quad$ {a product is negative iff one factor is positive and the other is negative: $(ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)$}

$\quad (x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0)$

$\equiv \quad$ {simplify both disjuncts}

$\quad (x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2)$

$\equiv \quad$ {the right disjunct is false for each x}

$\quad (-2 < x < 0) \vee F$

$\equiv \quad \{p \vee F \equiv p\}$

$\quad -2 < x < 0$

$\ldots \quad (x = 0 \vee x = -2) \vee (-2 < x < 0)$

$\equiv \quad$ {combine the conditions}

$$
\begin{array}{ll}
\bullet & x(2+x) < 0 \\
\equiv & \{\text{a product is negative iff one factor is positive and the other is} \\
& \text{negative: } (ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)\} \\
& (x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0) \\
\equiv & \{\text{simplify both disjuncts}\} \\
& (x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2) \\
\equiv & \{\text{the right disjunct is false for each } x\} \\
& (-2 < x < 0) \vee F \\
\equiv & \{p \vee F \equiv p\} \\
& -2 < x < 0
\end{array}
$$

$$
\begin{array}{ll}
\ldots & (x = 0 \vee x = -2) \vee (-2 < x < 0) \\
\equiv & \{\text{combine the conditions}\} \\
& -2 \leq x \leq 0
\end{array}
$$

$$
\begin{array}{ll}
\bullet & x(2+x) < 0 \\
\equiv & \{\text{a product is negative iff one factor is positive and the other is} \\
& \text{negative: } (ab < 0) \equiv (a < 0 \wedge b > 0) \vee (a > 0 \wedge b < 0)\} \\
& (x < 0 \wedge 2 + x > 0) \vee (x > 0 \wedge 2 + x < 0) \\
\equiv & \{\text{simplify both disjuncts}\} \\
& (x < 0 \wedge x > -2) \vee (x > 0 \wedge x < -2) \\
\equiv & \{\text{the right disjunct is false for each } x\} \\
& (-2 < x < 0) \vee F \\
\equiv & \{p \vee F \equiv p\} \\
& -2 < x < 0
\end{array}
$$

$$
\begin{array}{ll}
\ldots & (x = 0 \vee x = -2) \vee (-2 < x < 0) \\
\equiv & \{\text{combine the conditions}\} \\
& -2 \leq x \leq 0
\end{array}
$$

$\square$