Visual introduction to models with delay

Eliza Bánhegyi, János Karsai

Bolyai Institute, University of Szeged, Faculty of Science and Informatics, Hungary

Introduction

Motivation

- Phenomena in sciences (biology, physics, engineering…)
- delay models commonly used for describing several aspects of epidemiology
- appear in physiology (respiratory system, tumor growth, neural networks)

Didactic problems

- mainly applied (biology, chemistry, …) students concerned
- forecast the effect or result of an experiment, but …
- hardly know its mathamatical background
- visualize the process without precisely having all the necessary theory

What is a delay differential equation?

$$x'(t) = f(t, x(t-\tau_1), \dots, x(t-\tau_n)), t \geq t_0$$
$$x(t) = x_0(t), t \leq t_0$$

Some differences between ODE's and DDE's

- initial functions instead of initial values (infinite dimensional space!)
- **solutions form very complex semiflow of infinite dimension**
- solutions may not be differentiable, may not be continued backwards
- infinitely many equilibrium may exist, e.g. $x'^{(t)} = x(t) - x(t-\tau)$
- question of stability is a bit more complicated

Simple case-elimination

$x'(t) = -a\,x(t), a \in R$

$x(0) = x_0$, initial value

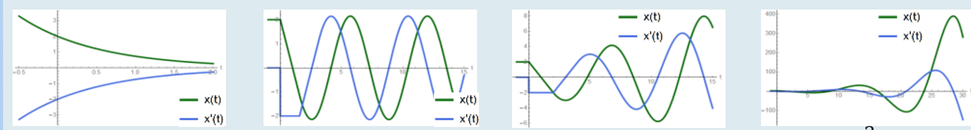
General solution: $x(t) = x(0)\,e^{at}$

Delay appears:

$x'(t) = -a\,x(t-\tau)$

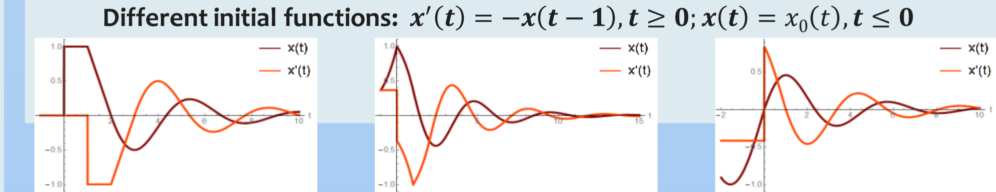
$x(t) = x_0(t), t \leq 0$, initial function

Different delays: $x'(t) = -x(t-\tau), t \geq 0; x(t) \equiv 2, t \leq 0$



$\tau = 0$ $\quad \tau = \frac{\pi}{2}$ $\quad \tau = 2$ $\quad \tau = \frac{3\pi}{2}$

Different initial functions: $x'(t) = -x(t-1), t \geq 0; x(t) = x_0(t), t \leq 0$



$x_0(t)$= UnitStep(t) $\quad x_0(t) = e^t$ $\quad x_0(t) = \sin(t)$

Linear equations

The general form of a scalar linear DDE is

$$x'(t) = Ax(t) + Bx(t-\tau)$$

We look for solutions in the form $x(t) \propto e^{\lambda t}$ as in the ordinary case. By substituting into the equation we obtain the characteristic equation

$$\lambda = A + Be^{-\lambda\tau}.$$

Observe:

- countably but infinitely many characteristic roots
- characteristic roots have no finite accumulation point

Stability theorem in terms of characteristic roots

- If all roots of the characteristic equation have negative real parts, then the equlibrium is asymptotically stable.
- If among the roots of charcteristic equation there is even one root with positive real part, then the equlibrium is unstable.

… in terms of coefficients

The following hold for $x'(t) = Ax(t) + Bx(t-\tau)$:

(a) If $A + B > 0$, then x = 0 is unstable.

(b) If $A + B < 0$ and $B \geq A$, then $x = 0$ is asymptotically stable.

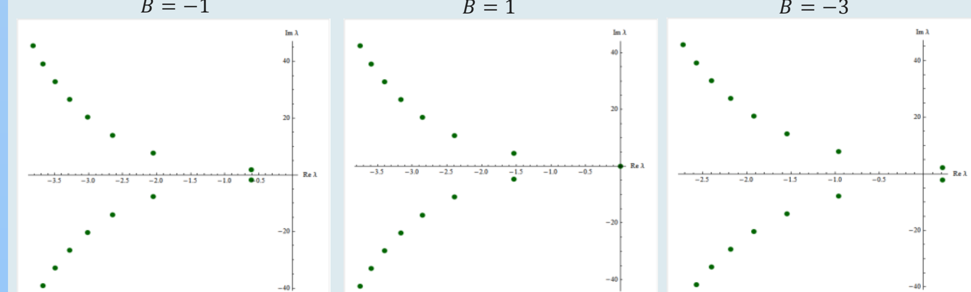
(c) If $A + B < 0$ and $B < A$, then there exists $\tau_c > 0$ such that $x = 0$ is asymptotically stable for $0 < \tau < \tau_c$ and unstable for $\tau > \tau_c$.

In case (c), there exist a pair of purely imaginary roots at $\tau_c = \frac{1}{\sqrt{B^2 - A^2}} \arccos(-\frac{A}{B})$.

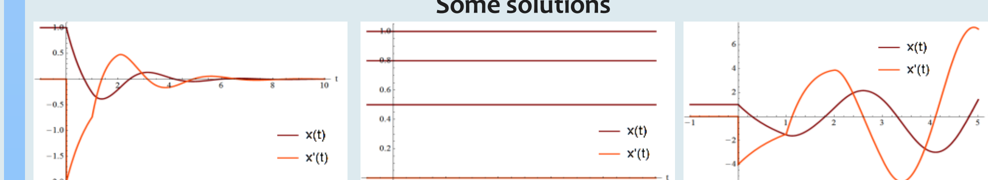
$$x'(t) = Ax(t) + Bx(t-\tau)$$

Characteristic roots

Asymptotically stable $A = -1$, $B = -1$ $\quad$??? $A = -1$, $B = 1$ $\quad$ Unstable $A = -1$, $B = -3$



Some solutions



Nonlinear equations

Linearization, why?

- often impossible to write down explicit solutions of a nonlinear DE
- exception: equilibrium solutions (the most important solutions)
- linearization: determine the behavior of solutions around equilibrium points

Stablity by linearization

- consider $x'(t) = f(x(t))$, x_0 is an equlibrium
- idea: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$
- take small perturbations about x_0, $u = x - x_0$ where $|u|$ is small
- approximate f with its multivariate Taylor series around x_0

Linearized ODE : $u'(t) = A\,u(t)$

Analogous DDE : $u'(t) = Au(t) + Bu(t-\tau)$

The solution of the linearized equation around 0 locally assembles to the solution of the nonlinear one around the equlibrium (see in [3]).

Example: a nonlinear model in biology- Cheyne-Stoke respiration

- human respiratory ailment
- alteration in the regular breathing pattern

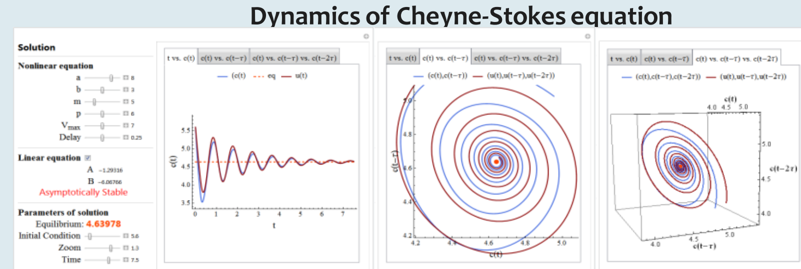
Physiological facts:

- $c(t)$: level of arterial CO_2
- τ: delay, CO_2 - sensitive receptors are situated in the brainstem however the gas-changing process happens in the lungs
- p: metabolic production of CO_2
- $V_{max}\frac{c^m(t-\tau)}{a^m + c^m(t-\tau)}$: ventilation response curve, described by Hill function
- a, b, m: positive parameters

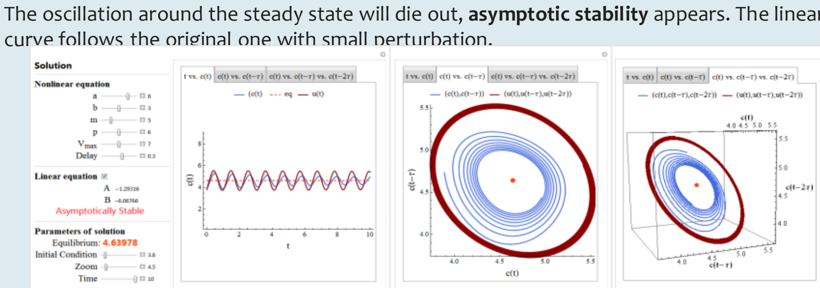
The dynamics of the arterial CO_2 level is then modelled [2] by

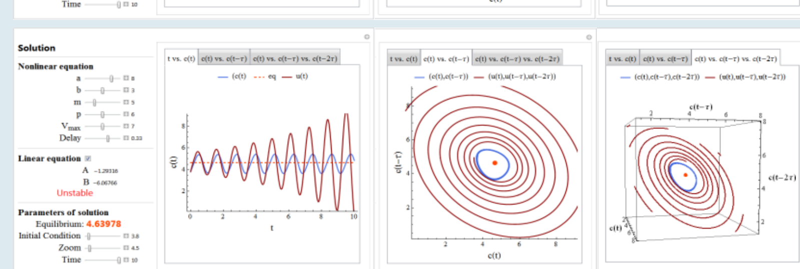
$$c'(t) = p - bV_{max}c(t)\frac{c^m(t-\tau)}{a^m + c^m(t-\tau)}$$

Dynamics of Cheyne-Stokes equation



The oscillation around the steady state will die out, **asymptotic stability** appears. The linearized curve follows the original one with small perturbation.





Exceeding the critical delay, periodic solution appears in the nonlinear system. The curves start strictly together then they separate from each other. The linear system is **unstable.**

References:

1. E. Bánhegyi: On a delay case of Cheyne-Stokes respiration
2. J.D.Murray: Mathematical Biology
3. H. Smith: An Introduction to Delay Differential Equations with Applications to the Life Sciences
4. Hirsch, Smale, Devaney: Differential Equations, Dynamical Systems, and an Introduction to Chaos