# A dynamic indroduction of fractional calculus


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## Introduction

### Then – and now

In the end of the 17th century L'Hospital asked Leibniz about the sense of :

$$\frac{D^n}{Dx^n},\quad n=\frac{1}{2}$$

i.e., the fractional derivative of order. Leibniz's answer was:

"An apparent paradox, from which one day useful consequences will be drawn."

Nowadays, the fractional - order calculus plays an important role in physical and other applications, as viscoelastic materials, fluid flow, diffusive transport, electrical networks, electromagnetic theory, probability and others.

Here a simple way of extension of the derivative and antiderivative to fractional order will be presented.

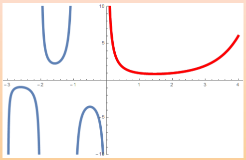
### A motivating example: the Gamma function

Task: Extend $n! := 1\,2\,3 \dots n$ to $x \in \mathbb{R}\ (x \geq 0)$

Idea: Use integration by parts: $\int_0^\infty e^{-t} t^n dt = \dots = n!$

Result: a natural extension of n!: $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$

i.e., the Gamma function, which gives $\Gamma(n) = (n-1)!$



## The concept of fractional integral by Riemann-Liouville

### Iterate the antiderivative

$$I^0(f)(t) = f(t);\ I^1(f)(t) = F(t) = \int_0^t f(\tau)d\tau;\ I^2(f)(t) = F(F(t)) = \int_0^t\int_0^\tau f(s)\,ds\,d\tau \dots$$

Using the Cauchy formula for the n$^{th}$ integral

$$I^n(f)(t) := F(F(\dots F(x)) = \int_0^t\int_0^{\tau_{n-1}}\dots\int_0^{\tau_1} f(\tau)\,d\tau\,d\tau_1 \dots d\tau_{n-1} = \frac{1}{(n-1)!}\int_0^t (t-\tau)^{n-1} f(\tau)\,d\tau$$

### Remark: Convolution

The convolution between two funtions $f, g : [0,\infty) \to R$ can be given as:

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau)\,d\tau$$

**Simple properties**

Distributive: $f*(g+h) = f*g + f*h$

Commutative: $f*g = g*f$

Associative: $f*(g*h) = (f*g)*h$

### Definition of fractional intergal

The positive integer $n$ can be replaced by any $\alpha > 0$.

$$\boxed{I^\alpha(f)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau}$$

**Properties:**

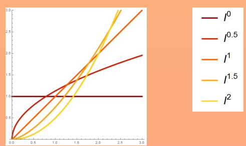
(1) $I^0(f)(t) = f(t)$

(2) $I^\alpha I^\beta = I^{\alpha+\beta} = I^\beta I^\alpha$

(3) $\alpha = 1$ : classical integral

**A simple example: *f* (*t*) = 1**

$$I^\alpha(1)(t) = \frac{t^\alpha}{\Gamma(1+\alpha)},\quad \text{if } \alpha \geq 0, t \geq 0$$



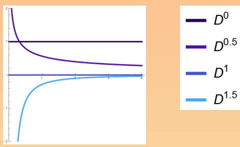

## Fractional derivative

Use the fractional integral to obtain the derivative of order α > 0: let $m$ be the ceiling of α, i.e., the intger for which m-1<α<m. Then, differentiate m times the fractional integral $I^{m-\alpha}$ .

$$D^\alpha f(t) := \begin{cases} \dfrac{d^m}{dt^m}\left[\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)\,d\tau\right], & \text{for } m-1<\alpha<m \\ \dfrac{d^m}{dt^m} f(t) & \text{, for } \alpha = m \end{cases}$$

**Consider *f* (*t*) = 1 again:**

$$D^\alpha(1)(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)},\quad \text{if } \alpha \geq 0, t \geq 0$$



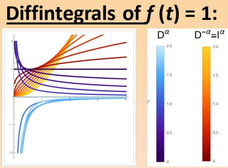

## Unified definition of Differintegrals

### The Riemann-Liouville differintegral

Now, join the above definitions: Let $\alpha \in \mathbf{R}$:

$$D^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau$$

**Diffintegrals of *f* (*t*) = 1:**

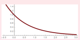


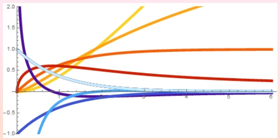

## More examples

Here one can see the diffintegrals of two well known functions. Colors are legended above. Formulas are somewhat „ugly", hence missed.

***f*** **(*t*) = e$^{-t}$**

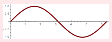
α=-2 … 2, step=0.5





***f*** **(*t*) = sin(*t*)**

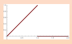
α=-2 … 2, step=0.5

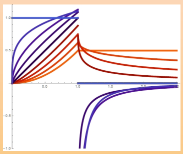


Some technical functions:

$$f(t) = \begin{cases} t & \text{for } 0<t\leq 1 \\ 0 & \text{otherwise} \end{cases}$$

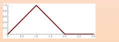
α=-1 … 1, step=0.25

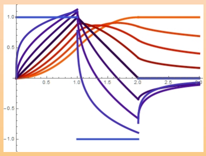




$$f(t) = \begin{cases} -|t-1|+1 & \text{for } 0<t\leq 2 \\ 0 & \text{otherwise} \end{cases}$$

α=-1 … 1, step=0.25





## Other definitions

The Riemann-Liouville definition can be applien to a wide class of functions, but the derivatives of the constant function is not zero. There are different ways of defining diffintegrals depending on applications and trying to resolve the problems of initial values. They can be applied to different classes of functions.

### Definition by Caputo

First derivate, then integrate:

$$D^\alpha(f)(t) = I^{n-\alpha}\left[\frac{d^n f(t)}{dt^n}\right]$$

Where $n$ is the ceiling of α

### Definition by Grünwald-Letnikov

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\dots$$

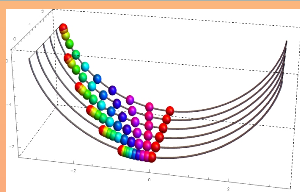
$$D^q f(x) = \lim_{h\to 0}\frac{(-1)^q}{h^q}\sum_{0\leq m<\infty}(-1)^m\binom{q}{m} f(x-mh)$$

## A historical application of fractional calculus

### The Tautochrone problem

The problem is to find a curve in the (x,y)-plane such that the time required for a particle to move down along the curve to its lowest point is independent of its initial placement on the curve. In deriving the known ordinary differential equation

$$\frac{dx}{dy} = \sqrt{\frac{2gT}{\pi^2 y} - 1}$$



The Caputo's diffintegral appears ${}^C D^{1/2}\sigma(y) = \dfrac{\sqrt{2g}}{\Gamma(\frac{1}{2})}T$ where σ is the length along the curve measured from the origin. See details in [1].

## Fractional differential equations

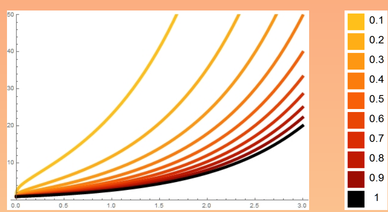
**1$^{st}$ and α$^{th}$ order linear equations**

$y'(t) = y(t)$

solution : $y(0) = 1,\quad y(t) = e^t$

$y^{(\alpha)}(t) = y(t),\quad 0<\alpha<1$

solution : $y(0) = 1,\quad y(t) = \sum_{i=0}^{\infty}\dfrac{t^{i\alpha}}{\Gamma(i\alpha+1)}$




**Linear oscillations with fractional terms**

Classical equation: $x'' + b\,x' + k^2 x = p(t)$

Some fractional equations form applications ($\alpha \leq 1$):

$x'' + b\,x' + k^2 I^\alpha(x') = p(t)$

$x'' + b\,D^\alpha(x) + k^2 x = p(t)$

$D^\alpha(x') + b\,x' + k^2 x = p(t)$

## Didactic summary - difficulties

- Real practical applications are motivating, but they need deep theories: differential equations, Complex analysis, Fourier and Laplace transform, Physics, Chemistry…
- How to introduce the concepts by only using the elementary analysis
- Difficulties in visualization and interpretation due to the lack direct geometric meaning (like area ortangent line)
- Which definition should be used first? What is the main difference between them?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