Playing Mathematics like a Chess Game?

An Educational View on Computer Theorem Proving

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CADGME Working Group
"Reasoning and Proving with Tool Support" (eduTPS'14)
Sept. 29, 2014 Halle, Germany

Outline

1. Preliminaries
   - Analogies: mathematics | chess
   - A new generation of mathematics SW
   - Learning chess by SW is exhaustive & fun !

2. Educational questions
   - Can we DO math without formality ?
   - How much formality needs to be taught ?
   - At which age which level of formality ?

3. The new SW generation can cover all mathematics, it works on mechanised formal logic and improves insight and confidence in mathematics.

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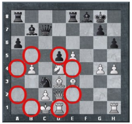
Analogies: mathematics | chess

Elements	
characters, e.g. . . . 26 letters, 10 numbers . . . $\forall\ \exists\ \wedge\ \vee\ \ldots\quad + - \cdot\ /\ \ldots$	64 fields, black and white 6 different kinds of figures 2 colours for figures
Rules	
for arranging characters in definitions, proofs, . . .	for moves of each figure changing between colours . . .
Compositionality	
to complex proofs . . . to extensive calculations	to arbitrary games to sophisticated strategies

$$1+\frac{2.(x+1)}{3.(x+1)}=\ldots\ldots$$



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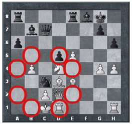

Analogies: mathematics | chess

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Rules	
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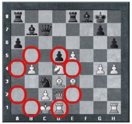

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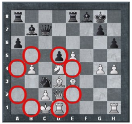

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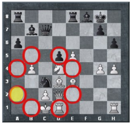
$$\boxed{a}+\boxed{b}=\boxed{b}+\boxed{a}$$

$$\boxed{1}+\frac{\boxed{2.(x+1)}}{\boxed{3.(x+1)}}=\frac{\boxed{2.(x+1)}}{\boxed{3.(x+1)}}+\boxed{1}$$



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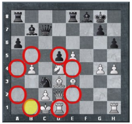

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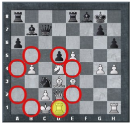
$$a.(b+c)=a.b+a.c$$

$$1+\frac{2.(x+1)}{3.(x+1)}=1+\frac{2.x+2.1}{3.x+3.1}$$



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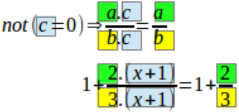

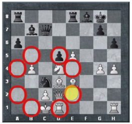

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Features of new generation:

1. check user input automatically, **flexibly** and reliably: Input establishes a *proof situation* (for *automated* proving) with respect to the logical context
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3.

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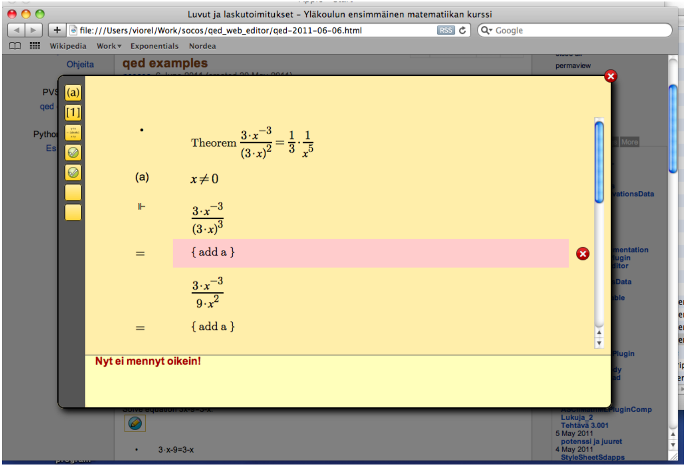
1

eMath by R.J.Back, Finland



Structured derivation editor with erroneous step marked.
See E-Math Project `http://emath.eu/`

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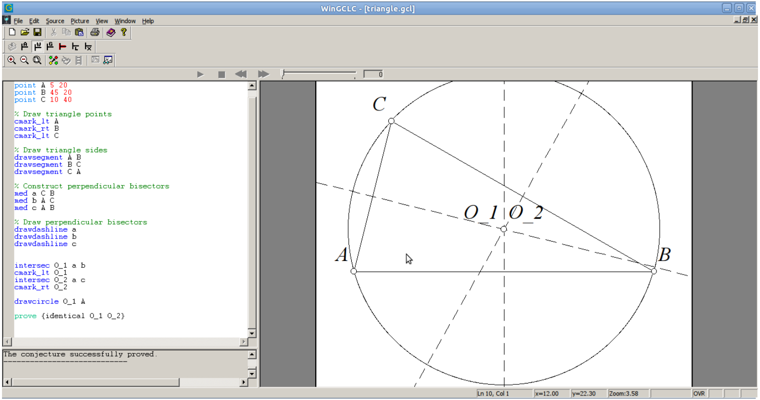
2

GCLC by P.Janicic, Serbia



GeoGebra's "academic relative", see *prove {identical O_1 O_2}* at bottom left.

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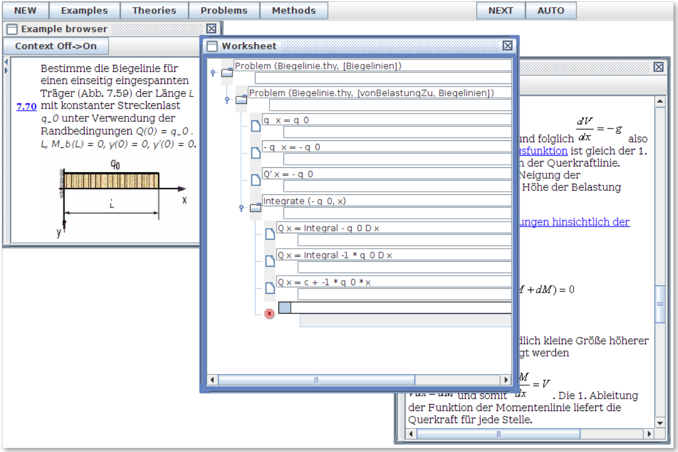
$\mathcal{ISAC}$ Austria



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The above three features . . .

- allow **step-wise solving math** problems in STEM [4]
- cover all math: formal specification, proof, algorithms,
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Learning chess by SW

Demo: interactive chess software (SW)

showing possibilities of learning

featured in an upcoming SW generation based on Computer Theorem Proving technology.

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DO math without formality ?

Chess: without rules (moves for knight etc) one can*not* play.

Math: the rule "*cancel products only*" $\frac{2\cdot\cancel{(x+1)}}{3\cdot\cancel{(x+1)}} = \frac{2}{3}$ (and $\xcancel{\frac{2+x}{3+x}}$)

...does it sufficie for $\frac{2\cdot\cancel{x}+1}{3\cdot\cancel{x}+1} = \frac{3}{4}$???

70% of STEM students can*not* reliably apply rules.

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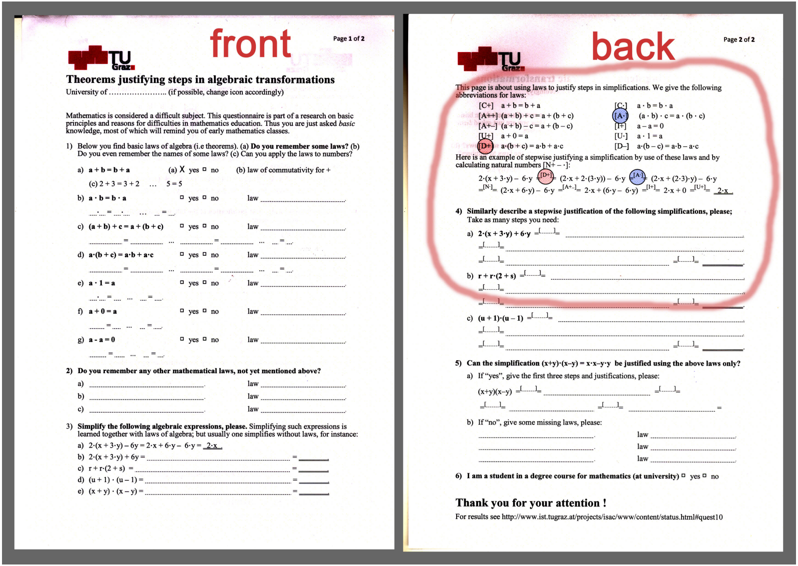

Page 1 of 2

TU Graz

Theorems justifying steps in algebraic transformations

University of .... (if possible, change icon accordingly)

Mathematics is considered a difficult subject. This questionnaire is part of a research on basic principles and reasons for difficulties in mathematics education. Thus you are just asked *basic* knowledge, most of which will remind you of early mathematics classes.

1) Below you find basic laws of algebra (i.e theorems). (a) **Do you remember some laws?** (b) Do you even remember the names of some laws? (c) Can you apply the laws to numbers?

a) **a + b = b + a** (a) X yes ▫ no (b) law of commutativity for +
(c) 2 + 3 = 3 + 2 ... 5 = 5

b) **a · b = b · a** ▫ yes ▫ no law ....
...·... = ...·... ... ... = ...

c) **(a + b) + c = a + (b + c)** ▫ yes ▫ no law ....
.... = ... = ... = ...

d) **a·(b + c) = a·b + a·c** ▫ yes ▫ no law ....
.... = ... = ... = ...

e) **a · 1 = a** ▫ yes ▫ no law ....
...·... = ... = ...

f) **a + 0 = a** ▫ yes ▫ no law ....
... = ... = ...

g) **a - a = 0** ▫ yes ▫ no law ....
... = ... = ...

2) **Do you remember any other mathematical laws, not yet mentioned above?**
a) .... law ....
b) .... law ....
c) .... law ....

3) **Simplify the following algebraic expressions, please.** Simplifying such expressions is learned together with laws of algebra; but usually one simplifies without laws, for instance:
a) 2·(x + 3·y) – 6y = 2·x + 6·y – 6·y = 2·x
b) 2·(x + 3·y) + 6y = ... = ...
c) r + r·(2 + s) = ... = ...
d) (u + 1) · (u – 1) = ... = ...
e) (x + y) · (x – y) = ... = ...

Page 2 of 2

TU Graz

This page is about using laws to justify steps in simplifications. We give the following abbreviations for laws:

[C+] a + b = b + a — [C·] a · b = b · a
[A++] (a + b) + c = a + (b + c) — [A·] (a · b) · c = a · (b · c)
[A+–] (a + b) – c = a + (b – c) — [I+] a – a = 0
[U+] a + 0 = a — [U·] a · 1 = a
[D+] a·(b + c) = a·b + a·c — [D–] a·(b – c) = a·b – a·c

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers [N+ – ·]:

2·(x + 3·y) – 6·y =[D+]= (2·x + 2·(3·y)) – 6·y =[A·]= (2·x + (2·3)·y) – 6·y
=[N·]= (2·x + 6·y) – 6·y =[A+–]= 2·x + (6·y – 6·y) =[I+]= 2·x + 0 =[U+]= 2·x

4) **Similarly describe a stepwise justification of the following simplifications, please;** Take as many steps you need:
a) **2·(x + 3·y) + 6·y** =[.....]= ....
=[.....]= ....
=[.....]= .... =[.....]= ....
b) **r + r·(2 + s)** =[.....]= ....
=[.....]= ....
=[.....]= .... =[.....]= ....
c) **(u + 1)·(u – 1)** =[.....]= ....
=[.....]= ....
=[.....]= .... =[.....]= ....

5) **Can the simplification (x+y)·(x–y) = x·x–y·y be justified using the above laws only?**
a) If "yes", give the first three steps and justifications, please:
(x+y)(x–y) =[.....]= .... =[.....]=
=[.....]= .... =[.....]= .... =
b) If "no", give some missing laws, please:
.... law ....
.... law ....
.... law ....

6) **I am a student in a degree course for mathematics (at university)** ▫ yes ▫ no

Thank you for your attention !

For results see http://www.ist.tugraz.at/projects/isac/www/content/status.html#quest10

Cannot apply rules

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[C+]	$a + b = b + a$	[C·]	$a \cdot b = b \cdot a$
[A++]	$(a + b) + c = a + (b + c)$	[A·]	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
[A+–]	$(a + b) - c = a + (b - c)$	[I+]	$a - a = 0$
[U+]	$a + 0 = a$	[U·]	$a \cdot 1 = a$
[D+]	$a \cdot (b + c) = a \cdot b + a \cdot c$	[D–]	$a \cdot (b - c) = a \cdot b - a \cdot c$

Here is an example of stepwise justifying a simplification by use of these laws and by calculating natural numbers [N+ – ·]:

$$2\cdot(x + 3\cdot y) - 6\cdot y \overset{[D+]}{=} (2\cdot x + 2\cdot(3\cdot y)) - 6\cdot y \overset{[A\cdot]}{=} (2\cdot x + (2\cdot 3)\cdot y) - 6\cdot y$$
$$\overset{[N\cdot]}{=} (2\cdot x + 6\cdot y) - 6\cdot y \overset{[A+-]}{=} 2\cdot x + (6\cdot y - 6\cdot y) \overset{[I+]}{=} 2\cdot x + 0 \overset{[U+]}{=} \underline{\underline{2\cdot x}}\,.$$

4) Similarly describe a stepwise justification of the following simplifications, please;
Take as many steps you need:

a) $\mathbf{2\cdot(x + 3\cdot y) + 6\cdot y}$ =[......]= ....

=[......]= ....

=[......]= .... =[......]= ________.

b) $\mathbf{r + r\cdot(2 + s)}$ =[......]= ....

=[......]= ....

Cannot apply rules 4

$2\cdot(x+3\cdot y) - 6\cdot y = (2\cdot x + 2\cdot(3\cdot y)) - 6\cdot y = (2\cdot x + (2\cdot 3)\cdot y) - 6\cdot y$
$=^{[N\cdot]}= (2\cdot x + 6\cdot y) - 6\cdot y =^{[A+-]}= 2\cdot x + (6\cdot y - 6\cdot y) =^{[I+]}= 2\cdot x + 0 =^{[U+]}= \underline{2\cdot x}.$

4) Similarly describe a stepwise justification of the following simplifications, please;
Take as many steps you need:

a) $\mathbf{2\cdot(x+3\cdot y)+6\cdot y} =^{[D+]}= (2\cdot x + 2\cdot(3y)) + 6y$
$=^{[A\cdot]}= (2\cdot x + (2\cdot 3)\cdot y) + 6y$
$=^{[N\cdot]}= (2\cdot x + 6\cdot y) + 6\cdot y =^{[A+]}= 2\cdot x + (6y + 6y) =^{[N+]}= \underline{2\cdot x + 12\cdot y}$

b) $\mathbf{r + r\cdot(2+s)} =^{[D+]}= r + r\cdot 2 + r\cdot s$
$=^{[D+]}= r(1+2) + r\cdot s$
$=^{[N+]}= r\cdot 3 + r\cdot s =^{[C\cdot]}= 3\cdot r + s\cdot r =^{[D+]}= \underline{(3+s)\cdot r}.$

c) $\mathbf{(u+1)\cdot(u-1)} =^{[D-]}= (u+1)\cdot u - (u+1)\cdot 1$
$=^{[C\cdot]}= u\cdot(u+1) - (u+1)\cdot 1$
$=^{[U\cdot]}= u\cdot(u+1) - (u+1) =^{[D+]}= u\cdot u + u\cdot 1 =^{[N-]}= \underline{u\cdot u - 1}.$
$- (u+1) =^{[U\cdot]}= u\cdot u + u - (u+1) =^{[N-]}= u\cdot u + u - u - 1$

5) Can the simplification $(x+y)\cdot(x-y) = x\cdot x - y\cdot y$ be justified using the above laws only?

a) If "yes", give the first three steps and justifications, please:

$(x+y)(x-y) =^{[D-]}= (x+y)\cdot x - (x+y)\cdot y =^{[C\cdot]}= x\cdot(x+y) - y\cdot(x+y)$
$=^{[D+]}= x\cdot x + x\cdot y - (y\cdot x + y\cdot y) =^{[A+]}= x\cdot x + (x\cdot y - y\cdot x + y\cdot y) =^{[C\cdot]}$

b) If "no", give some missing laws, please:

Cannot apply rules 5

$=^{[N\cdot]}= (2\cdot x + 6\cdot y) - 6\cdot y =^{[A+-]}= 2\cdot x + (6\cdot y - 6\cdot y) =^{[I+]}= 2\cdot x + 0 =^{[U+]}= \underline{2\cdot x}$

4) Begründen Sie bitte ebenso beim schrittweisen Vereinfachen; machen Sie soviele Schritte wie Sie brauchen:

a) $2\cdot(x + 3\cdot y) + 6\cdot y =^{[D^+]}=$ 2x + 6y + 6y = 2x + 12y
$=^{[C^+]}=$ 2x + 12y
$=^{[\ldots]}=$ $=^{[\ldots]}=$ 2x + 12y

b) $r + r\cdot(2 + s) =^{[D^+]}=$ r + 2r + 2·s
$=^{[\ldots]}=$ r + 2r + 2s
$=^{[A^{++}]}=$ 3r + 2s $=^{[\ldots]}=$ 3r + 2s

c) $(u + 1)\cdot(u - 1) =^{[A\cdot]}=$ u² + u − u − 1
$=^{[D^-]}=$ u² − 1
$=^{[\ldots]}=$ $=^{[\ldots]}=$ u² − 1

5) Lässt sich $(x+y)\cdot(x-y) = x\cdot x - y\cdot y$ *nur* mit obigen Gesetzen begründen ?

a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:

xy − xy − y² $(x+y)(x-y) =^{[D^-]}=$ x² + xy − xy − y² $=^{[A^{++}]}=$ x² − y² = x·x − y·y
$=^{[\ldots]}=$ $=^{[\ldots]}=$ =

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:

Can**not** apply rules 6

$=^{[N\cdot]}= (2\cdot x + 6\cdot y) - 6\cdot y =^{[A+-]}= 2\cdot x + (6\cdot y - 6\cdot y) =^{[I+]}= 2\cdot x + 0 =^{[U+]}= \underline{2\cdot x}$.

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$=^{[A^+]}= \underline{2x + 12y}$

$=^{[\ldots\ldots]}= \ldots\ldots \quad =^{[\ldots\ldots]}= \_\_\_\_\_\_$.

b) $r + r\cdot(2 + s) =^{[D^+]}= r + 2r + r\cdot s$

$=^{[A^+]}= 3r + r\cdot s$

$=^{[D^+]}=$ also $\underline{r\cdot(3+s)} \quad =^{[\ldots\ldots]}= \_\_\_\_\_\_$.

c) $(u + 1)\cdot(u - 1) =^{[D^-]}= u^2 - u + 1\cdot(u-1)$

$=^{[D^-]}= u^2 - u + \cancel{1}\, u - 1$

$=^{[1^+]}= \underline{u^2 - 1} \quad =^{[\ldots\ldots]}= \_\_\_\_\_\_$.

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a) Wenn "ja", geben Sie bitte die ersten drei Schritte samt Begründung an:

$(x+y)(x-y) =^{[D^-]}= x^2 - x\cdot y + y\cdot(x-y) =^{[D^-]}= x^2 - x\cdot y + x\cdot y - y^2$

$=^{[1^+]}= x^2 - y^2 =^{[\ldots\ldots]}= \underline{x\cdot x - y\cdot y} =$

b) Wenn "nein", geben Sie bitte einige der fehlenden Gesetze an:

Can**not** apply rules 7

$2\cdot(x+3\cdot y) - 6\cdot y \Leftarrow\Rightarrow (2\cdot x + 2\cdot(3\cdot y)) - 6\cdot y \Leftarrow\Rightarrow (2\cdot x + (2\cdot 3)\cdot y) - 6\cdot y$

$=^{[N\cdot]}= (2\cdot x + 6\cdot y) - 6\cdot y =^{[A+-]}= 2\cdot x + (6\cdot y - 6\cdot y) =^{[I+]}= 2\cdot x + 0 =^{[U+]}= \underline{2\cdot x}$

4) **Tomando la simplificación anterior como ejemplo, simplifica las siguientes expresiones algebraicas indicando en casa paso la regla usada.**
Usa tantos pasos como consideres necesarios:

a) $\mathbf{2\cdot(x+3\cdot y) + 6\cdot y}$ =[D+]= $2x + 6y + 6y$
=[D+]= $2x + 12y$
=[......]= .... =[......]= ....

b) $\mathbf{r + r\cdot(2+s)}$ =[D+]= $r + 2r + rs$
=[D+]= $3r + rs$
=[D+]= $(3+s)r$ =[......]= ....

c) $\mathbf{(u+1)\cdot(u-1)}$ =[......]= $u^2 - 1$
=[......]= ....
=[......]= .... =[......]= ....

5) **¿Puede la transformación (x + y) • (x-y) = x•x – y•y justificarse usando únicamente las reglas mencionadas más arriba?**

a) Si tu respuesta es “sí”, escribe los tres primeros pasos indicando la regla usada en cada paso:

$(x+y)(x-y)$ =[D+]= $(xx - xy) + (yx - y^2)$ =[I+]=
=[......]= $x^2 - y^2$ =[......]= .... =

DO math without formality ?

Chess: without rules (moves for knight etc) one can*not* play.

Math: the rule "*cancel products only*" $\frac{2\cdot\cancel{(x+1)}}{3\cdot\cancel{(x+1)}} = \frac{2}{3}$ (and $\cancel{\frac{2+\cancel{x}}{3+\cancel{x}}}$)

...does it sufficie for $\frac{2\cdot\cancel{x}+1}{3\cdot\cancel{x}+1} = \frac{3}{4}$???

70% of STEM students can*not* reliably apply rules.

Unreliable formal operation suffices

- for talking about math
- for standardised application of math
- *not* for inventive application of math
- *not* for creating new math knowledge.

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Outline

1. Preliminaries
   - Analogies: mathematics | chess
   - A new generation of mathematics SW
   - Learning chess by SW is exhaustive & fun !
2. Educational questions
   - Can we DO math without formality ?
   - How much formality needs to be taught ?
   - At which age which level of formality ?
3. The new SW generation can cover all mathematics, it works on mechanised formal logic and improves insight and confidence in mathematics.

How much formality taught ?

Chess: learn rules (moves etc) such than one can play.

Math: PISA's competence model …

1. Communication: *"… recognizing a problem situation … "*
2. Mathematising: *"… transform to a strictly mathematical form … "*
3. Representation: *"… using a variety of representations … "*
4. Reasoning: *"… logically rooted thought processes … "*
5. Strategies for solving problems: *"… critical control processes … "*
6. Using symbolic, formal and technical language and operations "… "
7. Using mathematical tools: *"… that may assist math activity … "*

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Which age which formality ?

Chess: Abstraction arises from experience with games.

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$$not\,(\boxed{c}=0) \Rightarrow \frac{\boxed{a}.\boxed{c}}{\boxed{b}.\boxed{c}} = \frac{\boxed{a}}{\boxed{b}}$$

$$1+\frac{\boxed{2}.\boxed{(x+1)}}{\boxed{3}.\boxed{(x+1)}} = 1+\frac{\boxed{2}}{\boxed{3}}$$

Matching rule with term creates a substition σ :

$\sigma = \{a \rightarrow 2, b \rightarrow 3, c \rightarrow (x-1)\}$

substitute σ *into rhs* : $\frac{a}{b} \rightarrow \frac{2}{3}$

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DOing math requires reliable formal operation

human intuition	$\longleftrightarrow$	mechanisation of thinking
creativity	$\longleftrightarrow$	logical rigor

STEM students need a formal base to trust math

common sense	$\longleftrightarrow$	syntax, formal logic
insight	$\longleftrightarrow$	tear to pieces

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individual maturity	$\longleftrightarrow$	curriculum
application of math	$\longleftrightarrow$	pure abstraction

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