Discussions induced by unexpected answers from a computer algebra system

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Background

- Computer Algebra Systems
  - Maple, Mathematica, Maxima, Axiom, Wiris …
  - are capable of solving many problems
    - more sophisticated
    - better
- Different users
  - professional use
  - educational purposes
    - different ways
- Different needs, different expectations
  - solvable problems
  - user interface
  - form of answers

Answers offered by CAS

- could be evaluated from different points of view
  - professional user's
    - mathematically correct
    - somewhat flexible output allowed (nuances do not confuse so much)
      - radians/degrees
    - …
  - student's
    - according to school mathematics
    - nuances could be important
    - …
  - …

=?

```
(%i2) solve(x^2=-1);
(%o2) [x = -%i, x = %i]
```

#	PROBLEM	Ax	De	Mc	Mp	Mm	Mu	Re
L6	derivative of above is 0 & above at 0 is 0	#	#	$\star^{17}$	$\star$	○	#	●
L7	$\sqrt{4r+4\sqrt{r+1}}$		○	$\star$	●	○	●	●
L8	$\sqrt{r}$	○				○	●	
L9	$\frac{z\pi}{2} - \pi^z\zeta(1-z) \Rightarrow 0$	—				●		
	M. Equations							
M1		●		●	●	○	●	●
M2	solve$(3x^3 - 18x^2 + 33x - 19 = 0, \mathbf{R})$	○	●	$\star$	$\star$	○	$\star$	●
M3	solve$(x^4 + x^3 + x^2 + x + 1 = 0)$	♮	●	●	●	○	●	●
M4	verify a solution of the above	●	●	●	●	●	$\star$	●
M5	solve$(x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23)$							
M6	solve$(x^7 - 1 = 0) \Rightarrow x = \{1, \{e^{\pm 2k\pi i/7}\}_{k=1}^{3}\}$	♮	$\star$	●	●	$\star$	♮	♮
M7	solve$(x^8 - 8x^7 + \cdots - 140x + 46 = 0)$			●	●	●	●	●
M8	solve$(e^{2x} + 2e^x + 1 = z, x)$							
M9	solve$(e^{2-x^2} = e^{-x}) \Rightarrow x = \{-1, 2\}[+$							
M10	solve$(e^x = x) \Rightarrow x = -W_n(-1)$ $(n$							
M11	solve$(x^x = x) \Rightarrow x = \{-1, 1\}$							

Michael Wester. *Computer Algebra Systems. A Practical Guide*. 1999

– 542 problems

- ● success! (hurrah!)
- # incompletely simplified, but some useful transformations were performed (groan)
- ε a surprising error occurred (ack!)
- ….

The unexpected answers

confusing and obstructive

or

opportunities (Paul Drijvers) and

a catalyst for rich mathematical discussion (Robyn Pierce, Kaye Stacey)

Opportunity, catalyst

Drijvers, P. (2002)

> *The first obstacle is: The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'. ... Recognizing equivalent expressions is a central issue in algebra, and still is when working in a computer algebra environment.*

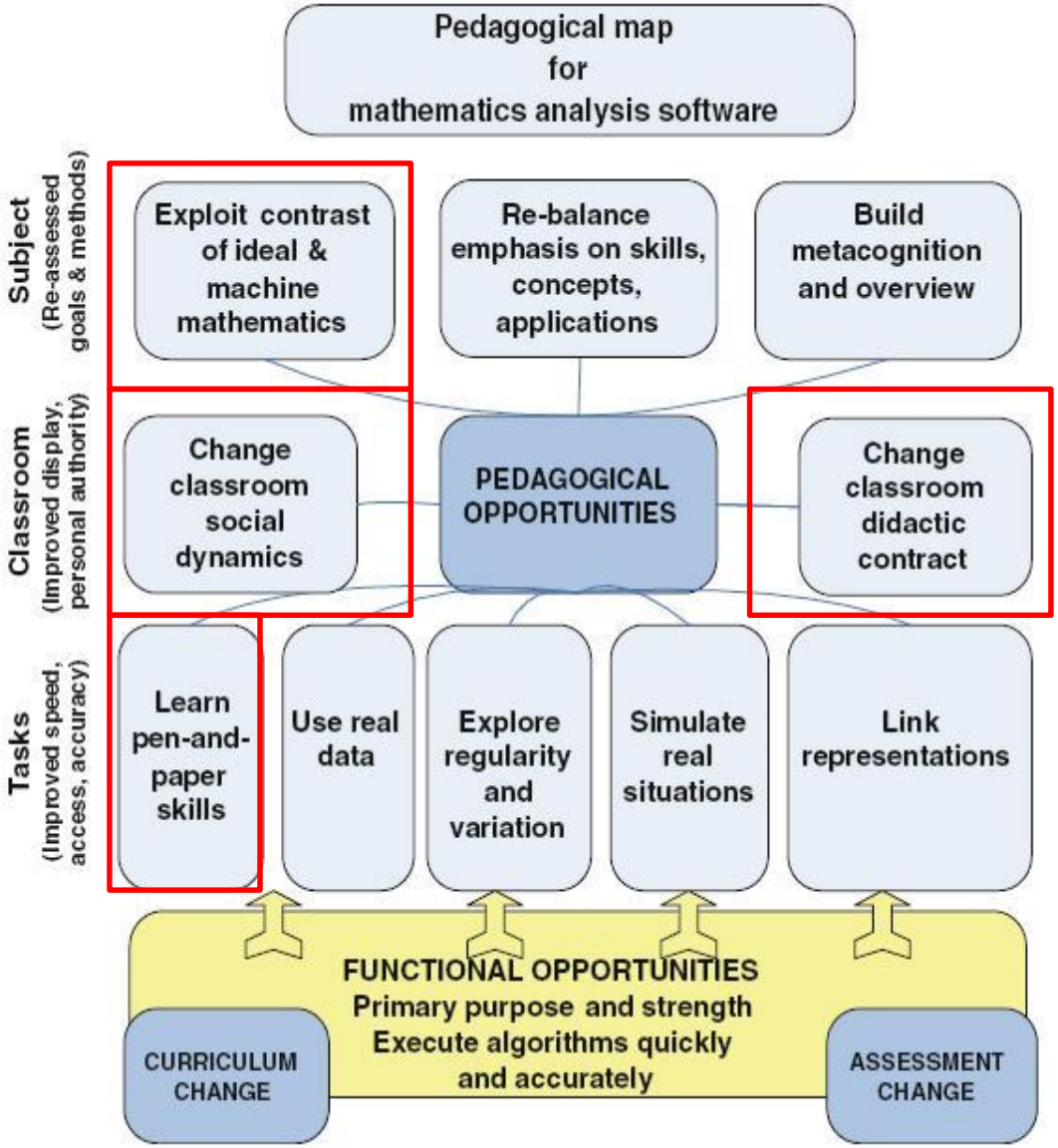
Pierce, R., & Stacey, K. (2010)

> *Unexpected mathematical results may be distracting and disheartening, but they are also pedagogical opportunities since they be used to provoke rich mathematical discussion*.

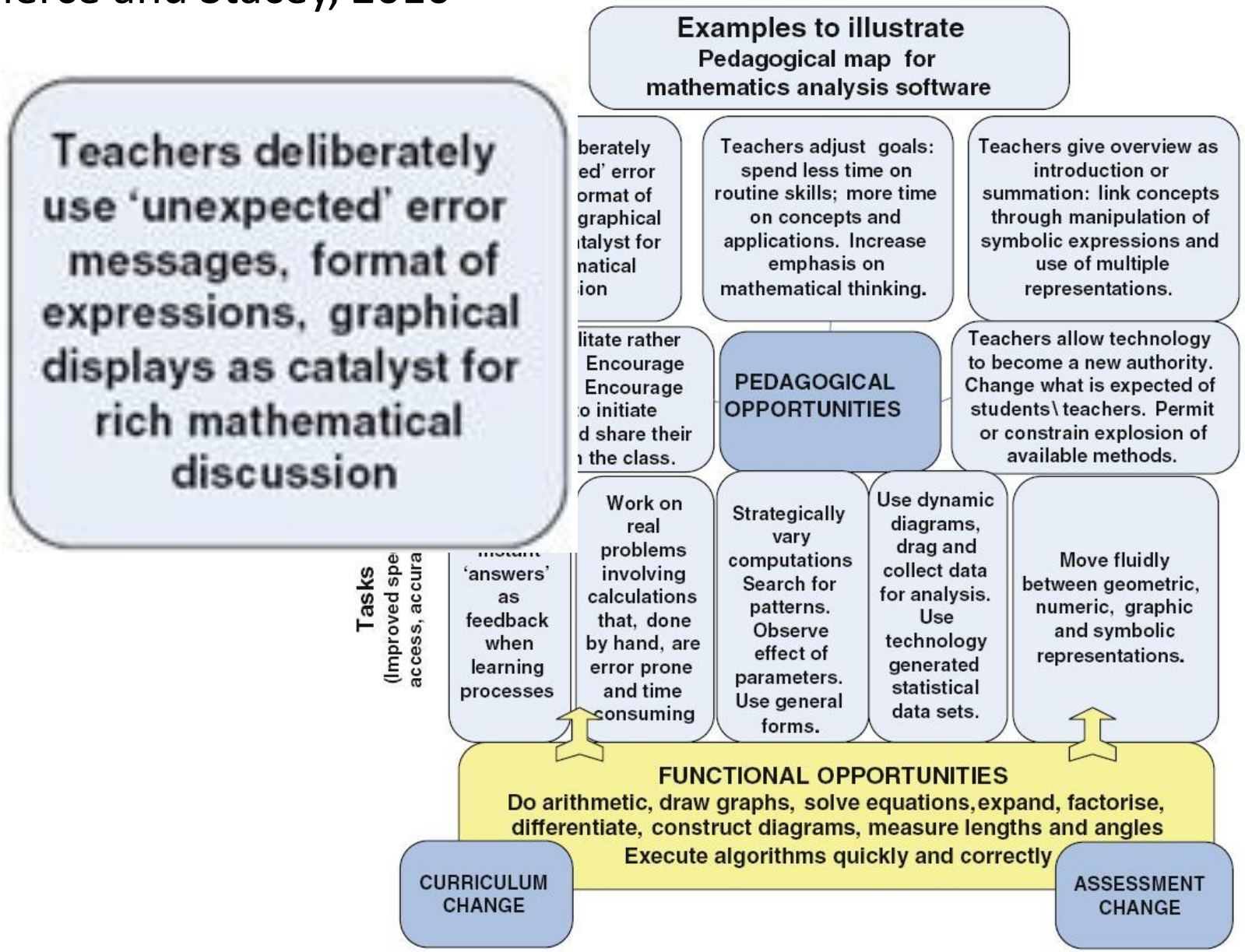
Buteau, C., Marshall, N., Jarvis, D. H., & Lavicza, Z. (2010)

> *Although practitioners have to deal with unusual or unexpected behaviour of CAS, this was occasionally shown to provide pedagogical opportunities*.

Pierce and Stacey, Mapping pedagogical opportunities provided by mathematics analysis software. *International Journal of Computers for Mathematical Learning*, *15*(1), 1-20. 2010



Pierce and Stacey, 2010



Rich mathematical discussion?

- Rather intuitive term
- Some papers
  - *The power of paper-folding tasks: Supporting multiplicative thinking and rich mathematical discussion*
  - *What's the Use of LOGO?*
  - *Modelling the cooling of coffee: Insights from a preliminary study in Indonesia*
  - *Using CAS to enrich the teaching and learning of mathematics*

- Mathematical discussion, discussion in math lessons …
  - probably quite rare??? (at least in Estonia)
  - not easy to provoke
  - suitable tasks

Discussion

- Whole class
  - Jennifer Jayne Ingram. *Whole class interaction in the mathematics classroom: A conversation analytic approach*
- Groups and pairs
  - teacher relinquishes some control
    - What language do students use?
      - Terminology
    - Are they still engaged with the task?
    - Are all of them active?

Pairs

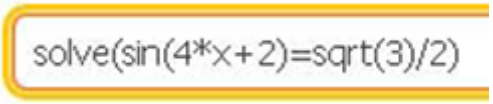
- Increase possibility to share their thoughts
  - practice to speak „mathematical language“
    - as in language classes
- Technically suitable
  - for recording

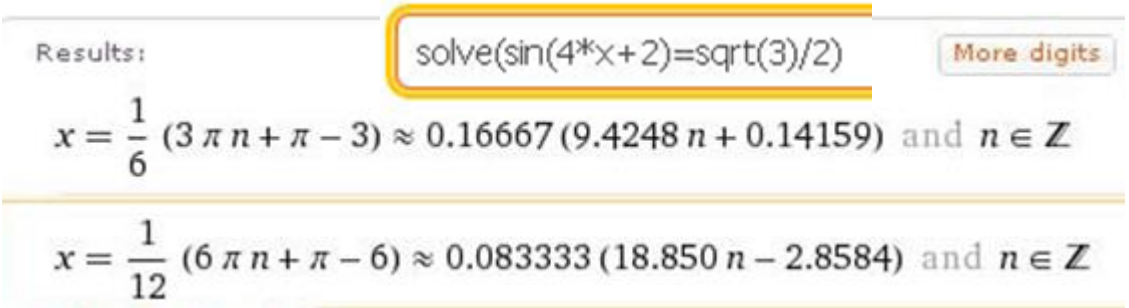
- Working in pairs in different courses at our university
  - for example, programming

- Possible problems
  - odd number of students
  - …

Student and CAS

- Student solves with CAS
- One-step
  - equation → answer





- Student compares his answers with the answers of a CAS

The broader perspectives

- in teaching-learning context
  - suggest a new method of using CAS for teaching and learning mathematics
    - students' discussion
    - critical thinking
    - deeper insight into important issues (such as equivalence)
- in research context
  - an analysis of students' worksheets and discussions could provide new opportunities for studying their thinking and learning
    - Anna Sfard:
      - possibility to analyse conversation as a microscope

Lessons

- First-year university students
- Course ”Elementary mathematics”
  - a somewhat repetitious course of school mathematics
- 90 minutes
    - an introduction
      - an overview of the lesson, the aims of the study
    - a period of equation-solving (ca 70 minutes)
    - closing (saving and copying data)
  - were taught by the first author (not a regular teacher of the course)

- Students in pairs (discussion!!!)
  - discussions were audio-taped
- The students
  - first solved an trigonometric equation (correctly or not) without CAS
  - then with a particular CAS
  - analyzed differences, equivalence and correctness **of their own answers and CAS answers**

Trigonometric equation

- Solving trigonometric equations
  - the variety of possible presentations of solutions
  - units of measurement
  - general and particular solutions

- Variety of their answers
  - several reasonable representations of the correct answer
  - different solution strategies, different-looking but still equivalent answers
  - different formulae in different regions
    - solution for $\sin x = m$

$$x = \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$
$$x = \pi - \arcsin m + 2n\pi,\ n \in \mathbb{Z}$$

    - or (as in Estonian textbooks, for example)

$$x = (-1)^n \arcsin m + n\pi,\ n \in \mathbb{Z}$$

iJMEST
International Journal of Mathematical Education in Science and Technology

How to 'check the result'? Discourse revisited

SERGEI ABRAMOVICH
School of Education and Professional Studies,

$$2 + \cos^2 2x = (2 - \sin^2 x)^2$$

Alan completed his solution with the formula

$$x = \pm \arcsin \frac{1}{\sqrt[4]{3}} + \pi n$$

ed the square root from both sides of equation (3), Betsy completed the formula

$$x = \pm \arccos\left(\pm\sqrt{\frac{3-\sqrt{3}}{3}}\right) + 2\pi n \qquad (4)$$

Christina's solution.

$$x = \pm\frac{1}{2}\arccos\frac{3-2\sqrt{3}}{3} + \pi n$$

Dave

$$x = \pm \arctan\sqrt{\frac{1+\sqrt{3}}{2}} + \pi n$$

Order

- The students had worksheets with equations and tasks
- The order of solvable equations
  - prescribed
- The students
  - first solved an equation (correctly or not) without a CAS
  - then with a particular CAS
- WolframAlpha (in the first three equations)
- A specific CAS was prescribed for the equation
  - the expected difference between the students' answers and the CAS answer
  - initiate an "intrigue", the effect of different representations

- Solve an equation $\sin(4x+2) = \frac{\sqrt{3}}{2}$ (without the computer at first).
  - How confident are you in the correctness of your answer?
- Solve the equation with the CAS WolframAlpha using the command solve.

  - How unexpected is the CAS answer at first view?
  - Analyze the accordance of your answer with the CAS answer! If you want to complement/correct your solution, please use the green pen.
  - What are the differences between your answer and the CAS answer?
  - How are your answer and the CAS answer related (analyze equivalence/nonequivalence, particular solutions/general solutions)?
  - Rate the correctness of your (possibly corrected) answer.
  - Rate the correctness of the CAS answer.
  - Rate the equivalence/non-equivalence of your (possibly corrected) and CAS answers.

$$\sin(4x + 2) = \frac{\sqrt{3}}{2}$$

$$2\sin 2x \cos 2x + \cos 2x = 0$$

in the interval [-30°; 0]

$$\frac{\tan^2 x}{\tan x} = 0$$

$$\tan\left(x + \frac{\pi}{4}\right) = 2\cot x - 1$$

$$\tan^3 x = \tan x$$

$$2\cos^2 x + 4\cos x = 3\sin^2 x$$

$$1 - \cos x = \sqrt{3}\sin x$$

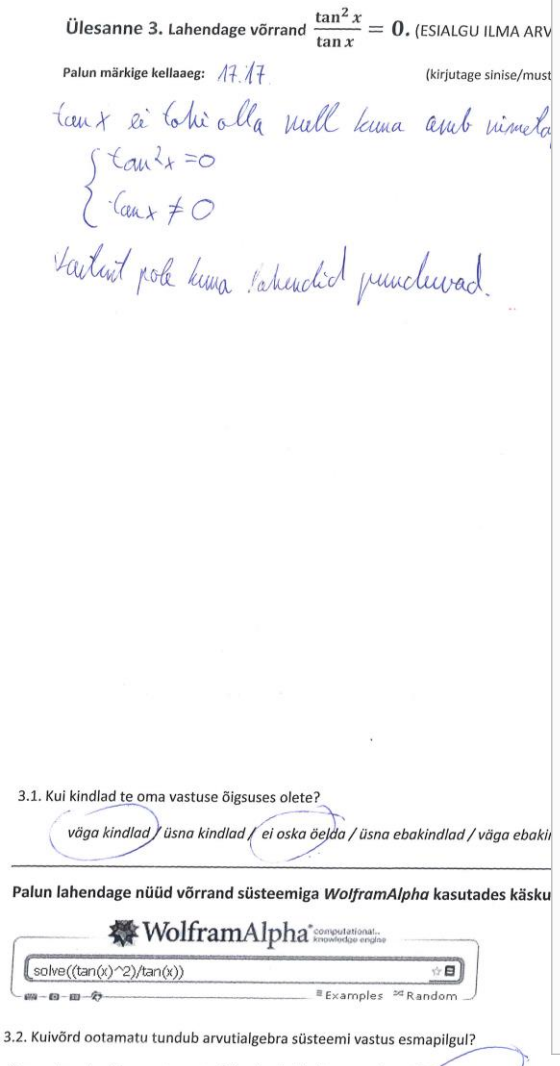
Collected data

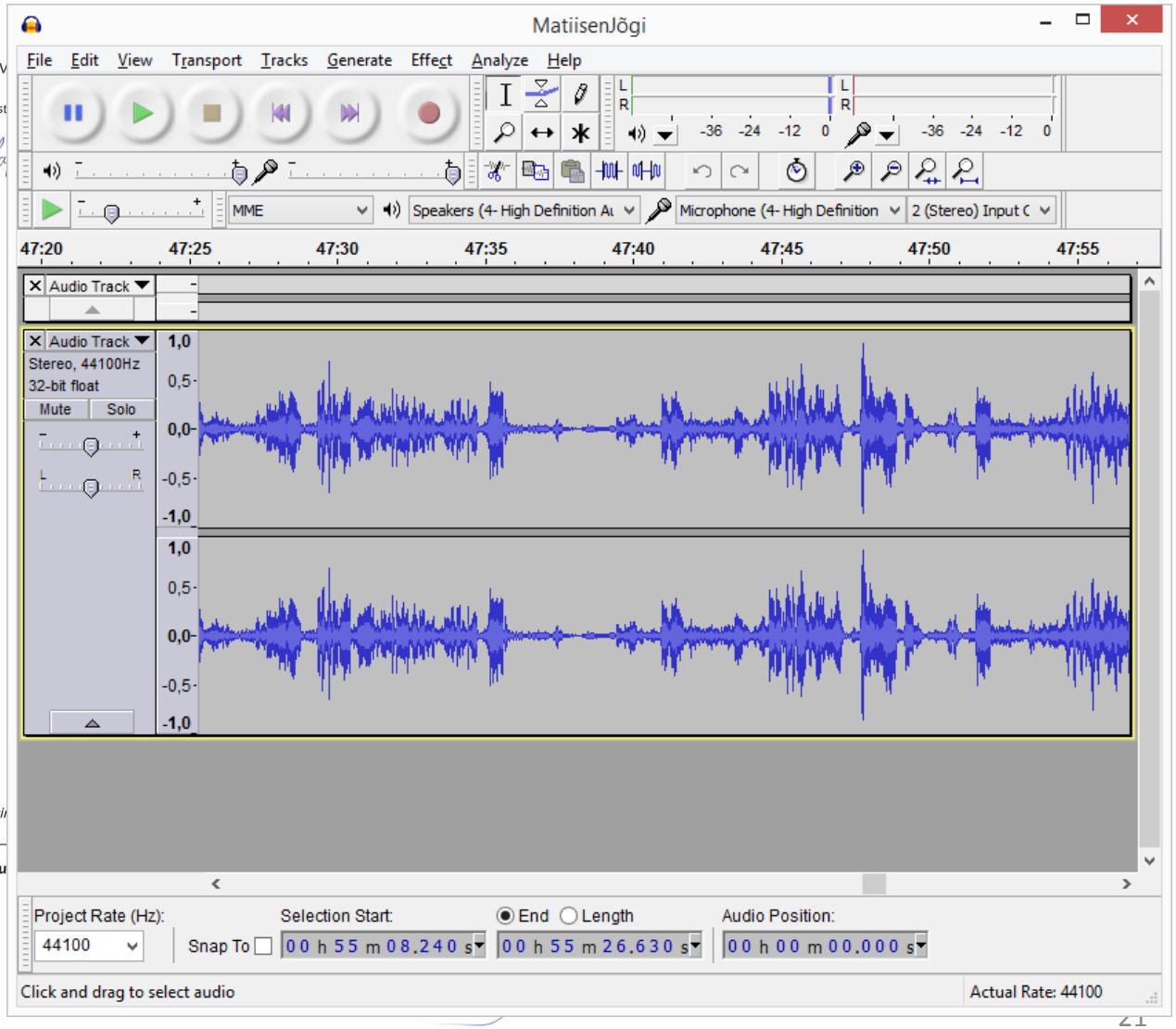
- Pairs
  - 29 in autumn 2012
  - 38 in autumn 2013
  - more than 200 episodes of solving the equation

- Worksheets
  - Papers
    - Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System, *Conferences on Intelligent Computer Mathematics: Mathematical Knowledge Management 2013.* CICM 2013. Published by Springer.
    - Students' Comparison of Their Trigonometric Answers with the Answers of a Computer Algebra System in Terms of Equivalence and Correctness, *International Conference on Technology and its Integration in Mathematics Education 2014.* TIME 2014. Submitted to the special issue.

- Audio recordings
  - for clarification of particular worksheet issues

Worksheets and recordings

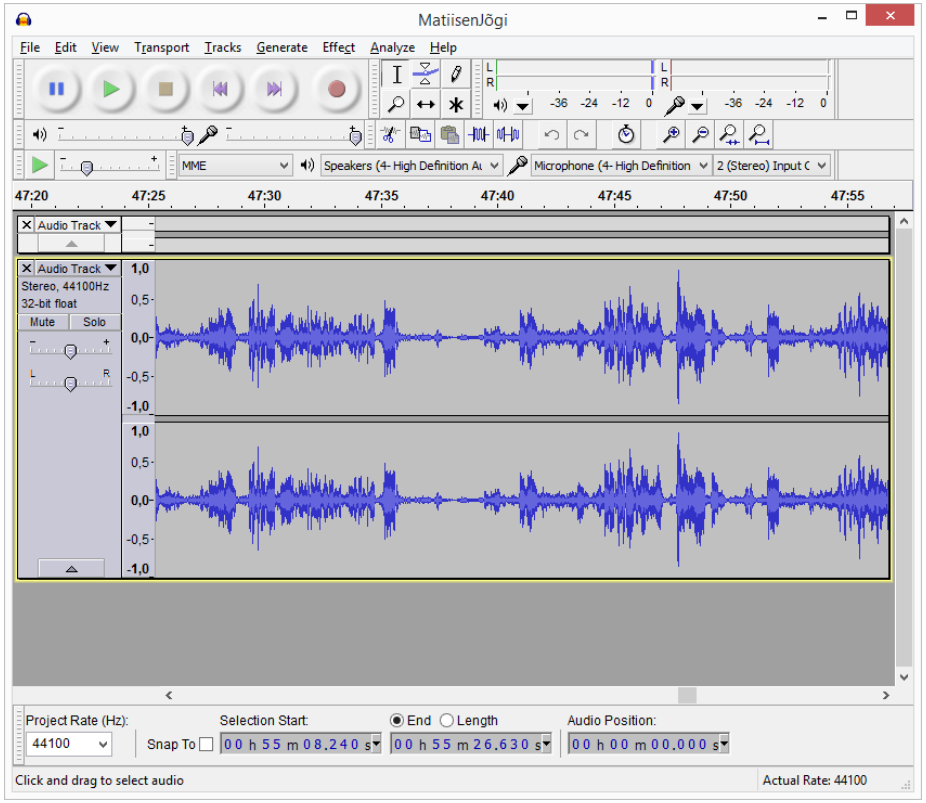




Kommentaarid!

Technique

- Laptop computers, computer lab
- Audacity
  - Audacity is free, open source, cross-platform software for recording and editing sounds.
- Problems
  - similar voices
  - noise
  - files
    - quite large
    - copying



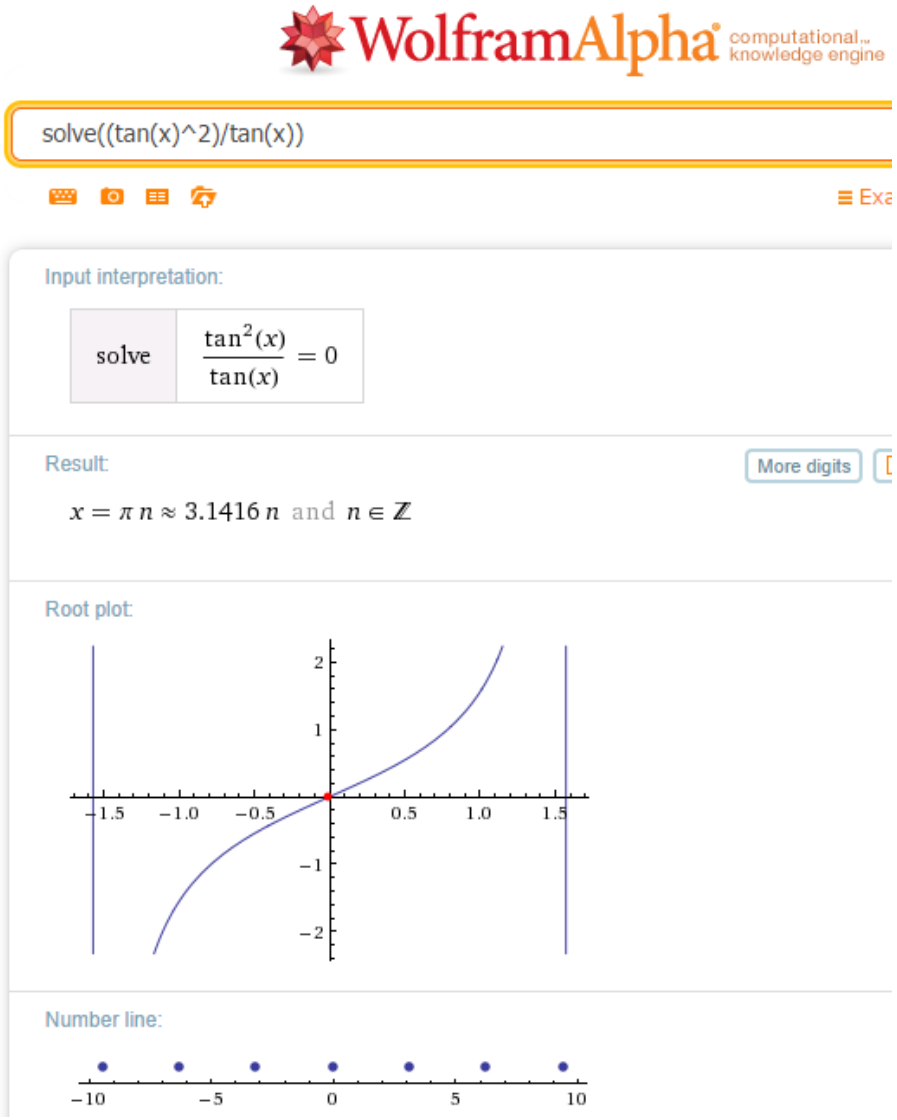
What may audio recordings offer?

- Anna Sfard:
  - possibility to analyse conversation as a microscope
- More elaborated overview about dialogue, conversation
- How to analyse? What topics?
  - Conversation analysis, Discourse analysis …
    - Sociology, anthropology, psychology, linguistics, education …
    - Sounds (intonation, etc.), gestures, syntax, the lexicon, style, rhetoric, meanings, speech acts, moves, strategies, turns …
- In mathematical context
  - lexicon
    - how mathematical?

What do they think about?

$$\frac{\tan^2 x}{\tan x} = 0$$

- No solution



Transcription

- Time consuming
  - 12 minute audio
  - ca 3 pages
  - ??? minutes of work.

43.34
A Tangens ruut x ja tangens. See on lihtne. Issand, kui lihtne see on. See on taandamine põhimõtteliselt ju. Tangens x ruut on tangens x korda tangens x. Jagad tangens x, null ja siis põhimõtteliselt kirjutama ainult. (Rõõmsalt)
B Tangens x ei tohi olla null.
A Tangens x ei tohi olla null või?
B Jah.
A Või … oota. Ei tangens x … null
B Tangens x ei saa olla null
A Et, x ei tohi olla null.
B See tangens. Võib-olla määramispiirkond.
A Hakkame siis pihta, kell on 17.17 selle kella järgi. Pane 17.16. Ei vahet pole. Siis … tangens x võib küll olla null.
[Mõtleb ilmselt tangensi määramispiirkonda hetkel, mitte selle võrrandi nimetajat.]
See on siis, kui x on null. Tangens x ei tohi olla 90. Oota. Kas me arvutame neid siit? [Ilmselt vaatasid valemite lehte.] X ei tohi olla 90 kraadi lihtsalt. Aga teeme selle ära. Selle õige asi on see, et on tangens x null lihtsalt. Oota. Jagajasse tuleb null. (Äratundvalt). See on võimatu. Jah. Tangens x ei tohi olla null, aga samal ajal tangens x võrdub null. Seega see ei ole võimalik. Õige, sul on õige. Sorry, mul läks väga aega, et läks kaua aega seda teha. Pane kirja see! Seepärast, et on määramispiirkond .. et kuna see on seal noh siis. Määramispiirkond on see, et tangens x ei tohi olla null. Kuna on nimetajas.
B Tangens ruut x võrdub nulliga ja tangens x ei võrdu nulliga. [Räägib, mida kirjutab]
A Kuidas me krdi astmemärki saame? Tead või? Oota, ma proovin nii.
B Sealt klaviatuurilt see väike märk või. Aga ma ei oska.

nine põhimõtteliselt ju. Tangens x ruut on tangens x korda tangens x. Jagad
x, null ja siis põhimõtteliselt kirjutama ainult. (Rõõmsalt)
ns x ei tohi olla null.
ns x ei tohi olla null või?

oota. Ei tangens x … null
ns x ei saa olla null
tohi olla null.
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ilmselt tangensi määramispiirkonda hetkel, mitte selle võrrandi lahendeid.]
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vaatasid valemite lehte.] X ei tohi olla 90 kraadi lihtsalt. Aga teeme selle ära.
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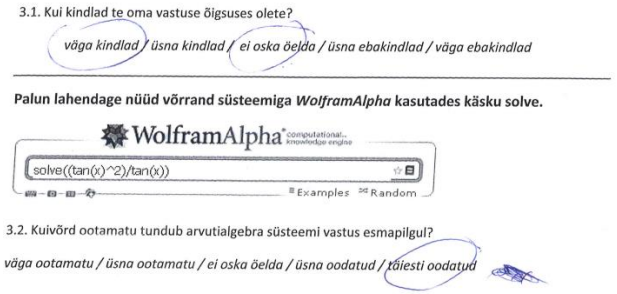
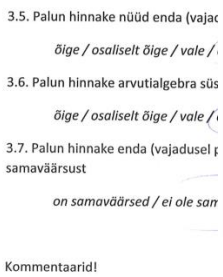
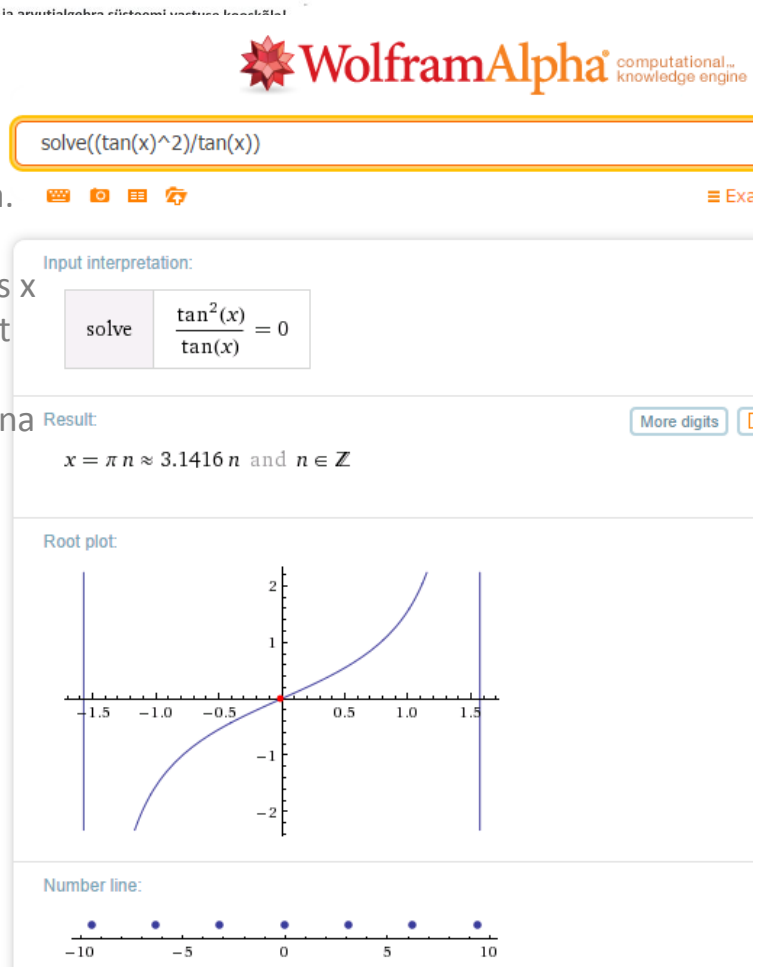
null. Seega see ei ole võimalik. Õige, sul on õige. Sorry, mul läks väga aega, et
a aega seda teha. Pane kirja see! Seepärast, et on määramispiirkond .. et
e on seal noh siis. Määramispiirkond on see, et tangens x ei tohi olla null. Kuna
tajas.
ns ruut x võrdub nulliga ja tangens x ei võrdu nulliga. [Räägib, mida kirjutab]
s me krdi astmemärki saame? Tead või? Oota, ma proovin nii.
klaviatuurilt see väike märk või. Aga ma ei oska.

Audio →transcription
Worksheet
CAS screenshot







Tangent may not be 90.

This is simple. Goodness, how simple!

Wait! Zero will be in the denominator [Student recognized]. It is impossible. Yes. Tangent x may not be 0, but tangent x is 0 in same time. Thus, it is not possible. Right! You are right. Sorry, it took time for me…

ee on lihtne. Issand, kui lihtne see on. Se
ruut on tangens x korda tangens x. Jagad tangens x, null ja siis põhimõt
B Tangens x ei tohi olla null.
A Tangens x ei tohi olla null või?
B Jah
A Või … oota. Ei tangens x … null
B Tangens x ei saa olla null

$$\frac{\tan^2 x}{\tan x} = 0$$

Cannot solve. Take the next task.

I am hungry. I want to eat.

How do we get the exponent symbol?

mispiirkond.
ahet pole. Siis
randi nimetaja
See on siis, kui x on null. Tangens utame neid siit? [Ilmselt vaatasid valemite lehte.] X
ei tohi olla 90 kraadi lihtsalt. Aga
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et on määramispiirkond . et kuna see on seal noh siis. Määramispiirkond on see, et n

There are no solutions. But it found them [a bit surprisingly]

Let's say this task was false. So, it is simpler.

ulliga ja tangens x ei võrdu nulliga. [Räägib, mida kirjutab]
rki saame? Tead või? Oota, ma proovin nii.
B Sealt klaviatuurilt see väike märk või. Aga ma ei oska.

B: It seems to be grey area.
A: It is grey area.

Tangent is a very weird thing. Actually, if x tends to 90 tangent tends to infinity

Conclusion

- The comparison of students' and CAS answers seems to work
  - they were involved with trigonometry whole lesson
- The task seemed to be new for the students
  - Usually, only the solution of an equation is needed and not more
  - The format seemed to be interesting and compelling
- These somewhat unexpected answers could support the discussion and provide a possibility to activate students
- In fact, the role of the teacher was mainly to introduce the lesson and to answer some questions during the lesson

Conclusion

- The students did not make enough use of the provided opportunity to analyse their own answers and the CAS answers in writing on worksheets
- The recordings give extra information
  - that could be interesting and useful
- Transcription is very time-consuming
  - reasonable if there are different aims
    - didactical
    - linguistics

Future

- Paper to December 15
- Autumn 2014
  - other equations

$$\sqrt{2x} = \sqrt{x-1}$$

$$x + \log x = \log x - 1$$

$$x + \frac{1}{x} = \frac{1}{x}$$