Structured derivations in practice: Experiences from the E-math project

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EUROPEAN UNION
EUROPEAN REGIONAL DEVELOPMENT FUND
INVESTING IN YOUR FUTURE

CENTRAL BALTIC
INTERREG IV A
PROGRAMME
2007–2013

Topics

- Structured derivations
- E-math project
- E-book technology for mathematics
- What after E-math

Structured derivations

New method for presenting mathematical arguments

Forces both teacher and student to explicitly justify each argumentation step in a derivation.

Hidden agenda: reintroduce proofs and exact reasoning in high school math

Easy to provide computer support for writing and analysing structured derivations

Works for all level of mathematics:from lower to upper secondary education and to universities

Example: Prove the conjugate rule:
$(a+b)\,(a-b) = a^2 - b^2$

- $(a+b)\,(a-b)$

justification

$=$ {distribution rule: $x(y-z) = xy - xz$}

$(a+b)\,a - (a+b)\,b$

$=$ {distribution rule: $(x+y)z = xz + yz$}

$a^2 + b\,a - a\,b - b^2$

$=$ {middle terms cancel each other}

$a^2 - b^2$

□

Fixed format makes it possible to check correctness of each derivation step with computers

Structured derivations

Electronic learning environment for mathematics

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TURKU ÅBO

Turun yliopisto
University of Turku

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E-Math project

- EU-project, May 2011 to December 2013
- Purpose: Improve student competence in mathematics
- Build a computer based environment for teaching and learning mathematics
- Teaching material is based on structured derivations
- Text books are interactive e-books
- Funding: Central Baltic Interreg IV A Programme 2007-2013

- Some 20 people in project team
- Total funding 2.3 milj euros

Partners

- Project partners:
  - City of Turku (lead partner)
  - City of Stockholm
  - City of Tallin
  - City of Mariehamn
  - Åbo Akademi University/IT Dept
  - University of Turku/IT Dept
- Universities have built the e-books and the text book content
- Cities have carried out pilot studies on using e-books

Main results

- A new software platform for interactive webbased e-books for mathematics
- New electronic textbooks for high school mathematics
- All textbooks based on structured derivations
- Educated teachers in structured derivations and e-books
- Piloted the use of e-books in i 15 highschools, in three countries
- Developed tools for authors to cooperate on writing e-books

New textbooks in mathematics

- Interactive e-books for first year studies in high school (K10)
- All textbooks fullfill the official curriculum requirements in Finland, Sweden and Estonia
- Textbooks in Finland (advanced math)
  - Functions and equations, Polynomials, Geometry, Analytic geometry
  - Available in both Finnish and Swedish
- Textbooks in Sweden (science curriculum)
  - Matematik 1C and Matematik 2C
- Textbooks in Estonia
  - Arvuhulgat. Avaldised. Võrrandit ja võrratused. Trigonomeetria. Vektor tasandil. Joone võrrand
- All books freely downloadable from the project pages
- Finnish books are multilingual
  - First book available in Finnish, Swedish, English, Estonian and Norwegian

Pilot studies

- Autumn 2012 and autumn 2013
- 15 high schools participated, appr. 1 000 students alltogether
- All autumn math courses taught with E-math books

Stockholm
Kärrtorps gymnasium
Södra Latin

Åbo
Aurajoen lukio
Kupittaan lukio
Luostarivuoren lukio
Raision lukio
Naantalin lukio
Loimaan lukio

Tallinn
Lasnamäe Gümnaasium
Pelgulinna Gümnaasium
Tallinna Kristiine Gümnaasium
Tallinna Kuristiku Gümnaasium
Tallinna Nõmme Gümnaasium
Tallinna Tehnikagümnaasium
Tallinna Tõnismäe Gümnaasium
Tallinna Ühisgümnaasium

Mariehamn
Ålands lyceum

Main findings

- Teachers like structured derivations A LOT, they are fans
- Students appreciate structured derivations:
  - it makes them understand their own and teachers solutions better
  - easier to find errors in their own solutions
  - but: it takes them longer to write down the solutions (the justifications take time)
- E-book is considered very good and modern
  - but some initial problems in starting to use the e-book
  - students are very interested in using computers in math education
  - most of them are quite good at it (they are “diginative”)
  - but not everybody
  - some teachers are also lacking in computer skills

E-math project: Student and teacher feedback from pilots

”...the way of calculating kind of helped, because you had to really think carefully about what to do. You got fewer calculation errors. At least I got. ”
Student, Finland

”I recommend this to anyone who is weak in mathematics. I was not good, my grades were 3, but now they are 5. Structured derivations helped me.”
- Student, Estonia

”Here you really learn to know what to do at each step, you cannot just jump to anything without knowing what you are doing, in each step you have to really know what you are doing.”
- Student, Finland

”I see who has done the homework. I still remember the day when all 18 students had sent me their solutions.”
– Teacher, Estonia

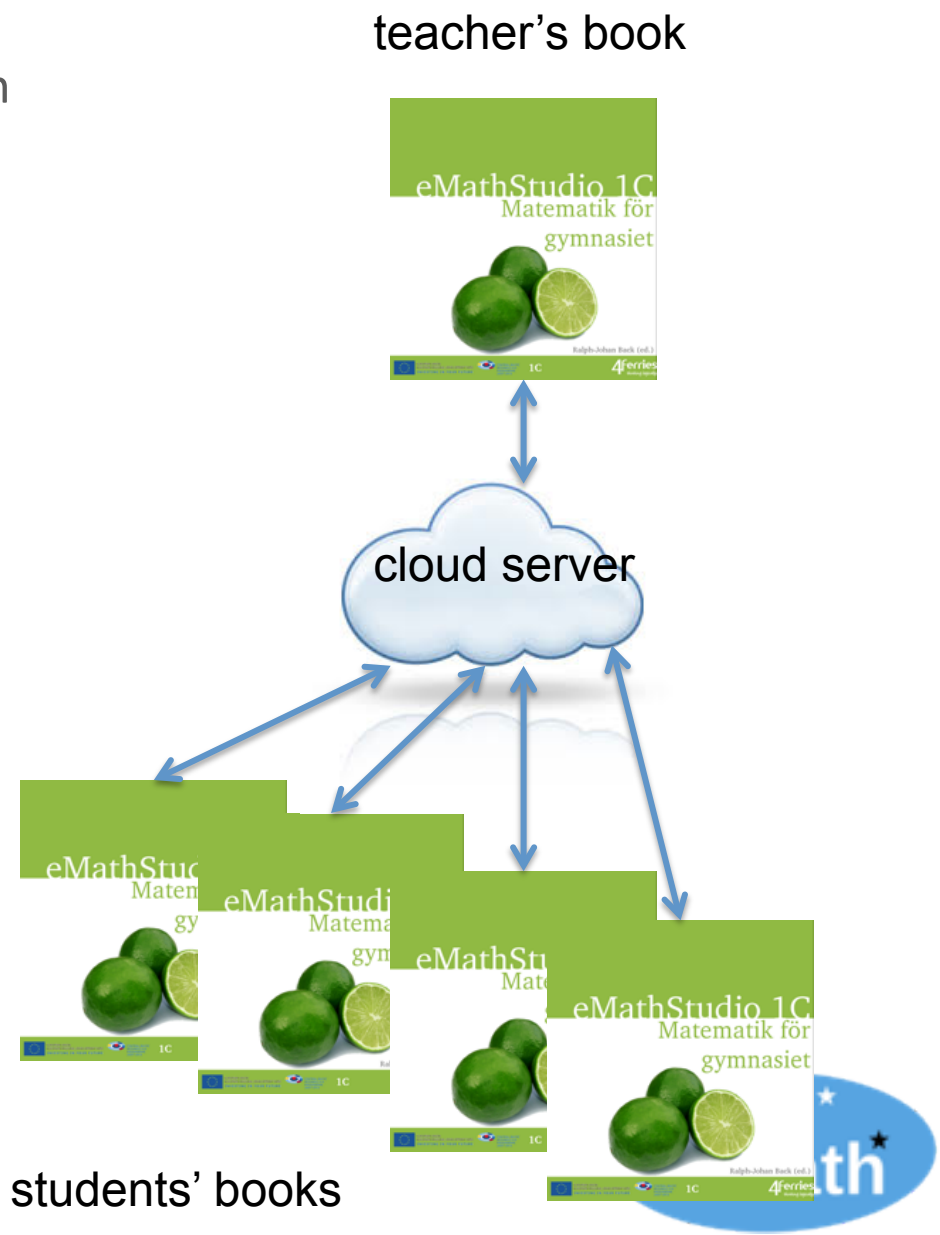
”I have been hoping that students could justify their solutions verbally, but now they write down their justifications. Which is even better.”
- Teacher, Estonia

The E-math learning environment

- Supports different scenarios
  - Traditional class room teaching
  - Virtual (net based) classes
  - Inverted class rooms
  - Individual tutoring of students
  - Self studies

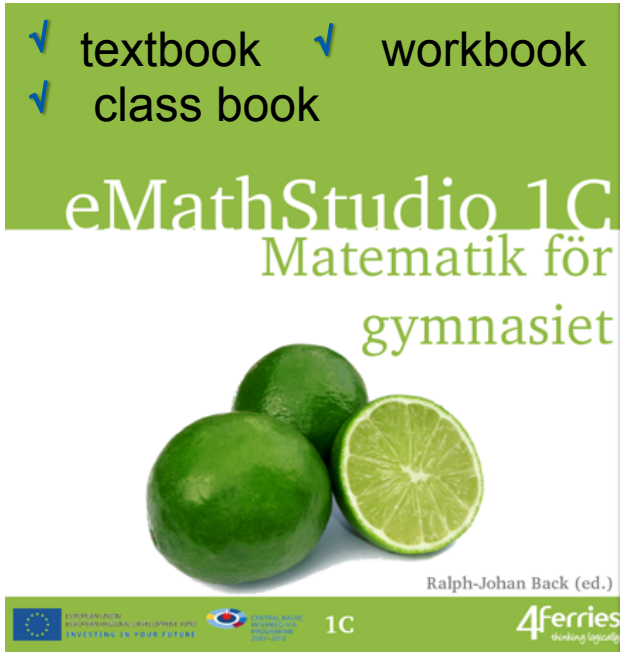
E-book architecture

- E-books in a class are connected to each other
- The e-books communicate via a cloud server
- Server keeps backup of each e-book
- Same e-book can be used on different computers
- e-book available on laptops and tablets (Window, Mac, Linux, Android, iOS9
- E-books can be used both online and offline
- Student's e-book can be used as ordinary textbooks without cloud service
- More functionality available when e-books are online



E-math e-books

Students' e-book consists of
- a textbook
- a workbook, and
- a classbook



Teachers' e-book also has
- a teacherbook

Textbook features

The e-book is a digital text book on some specific mathematical topic

textbook

Screenshot

The Existence of Solutions

The previous equation has exactly one solution. However, there are equations with no solutions. Consider for instance the equation $x + 2 = x$.

Example

- $x + 2 = x$

$\Leftrightarrow$ { subtracting x from both sides }

$x + 2 - x = x - x$

The result of the derivation is that

$$x + 2 = x \Leftrightarrow F$$

Explanation

The equation is false for every value of the variable x, in another words, there is no value for x that satisfies the this equation. We say that an equation like this is *identically false*.

First-Degree Equations

All the equations we have treated so far have been first-degree equations.

Theory 24: First–degree equation

A *first-degree equation* is an equation that can be written in the form

$$a \cdot x + b = 0,$$

where $a \neq 0$.

Theory

This type of equation has exactly one solution.

- Show that the equation $a \cdot x + b = 0$ has exactly one solution, when

(a) $a \neq 0$

$\Vdash$ $a \cdot x + b = 0$

$\Leftrightarrow$ { subtracting b from both sides and simplifying }

$a \cdot x = -b$

$\Leftrightarrow$ { dividing both sides by a, which is allowed since $a \neq 0$, and simplifying }

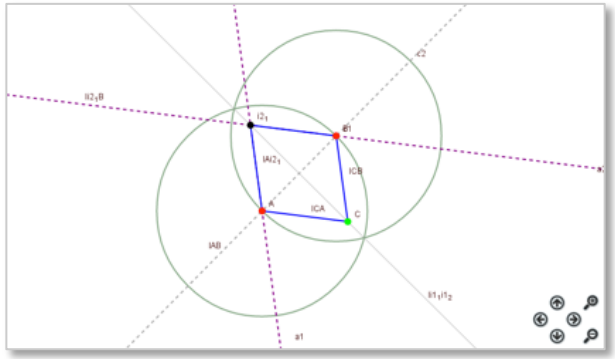
textbook

Interactive content

Illustrate concepts, theory and examples with

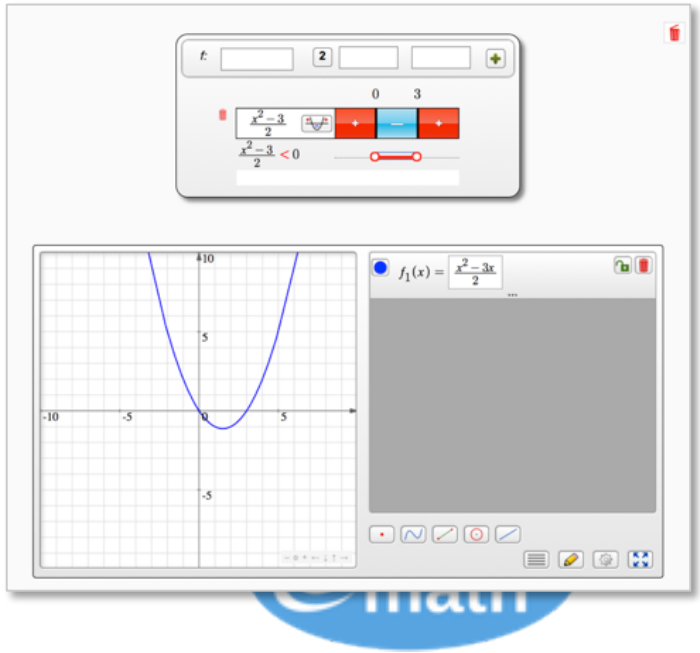
- interactive function graphs
- interactive figures
- interactive tables, etc

eMath Studio Interaction tools are specially made for mathematics education



move red points in geometric figure

change parameters in function graph



textbook

Interactive quizzes

Quizzes and problems for checking that students have undestood theory

Student's answers are checked automatically

Guides student into solving more demanding assignments on his/her own

fill in missing information in derivation

reorder steps in derivation

answer questions about function

Fyll i härledningen

- $\frac{2-x}{3} = \frac{3 \cdot x}{4}$

$\Leftrightarrow$ { ☐ }

$4 \cdot (2-x) = 3 \cdot 3 \cdot x$

$\Leftrightarrow$ { ☐ }

$8 - 4 \cdot x = 3 \cdot 3 \cdot x$

$\Leftrightarrow$ { ☐ }

Fyll i härledningen

- $\frac{2x-1}{3} = \frac{2x+1}{2}$

$\Leftrightarrow$ { Multiplicerar båda led med 6 }

$4x - 2 = 6x + 3$

$\Leftrightarrow$ { Förkortar }

☐

$\Leftrightarrow$ { Multiplicerar parenteserna }

☐

Ordna härledningen

$x = 12$

- $-x + 5 = \frac{22 - 3x}{2}$

$\Leftrightarrow$ { Slår ihop termerna }

$-2x = 22 - 10 - 3x$

$\Leftrightarrow$ { Adderar $3x$ till båda led }

$\Leftrightarrow$ { Subtraherar 10 från båd

$-2x + 3x = 22 - 10$

$\Leftrightarrow$ { Multiplicerar upp parente ledet och förkortar i det hö

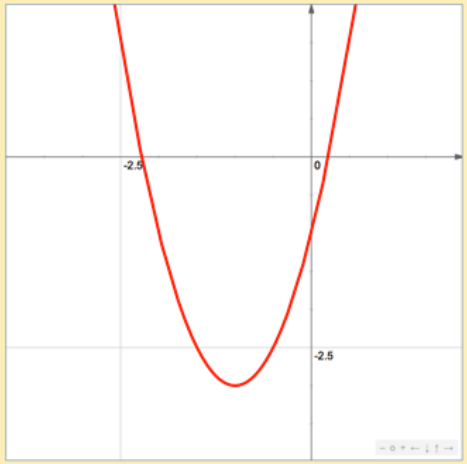
☐

$-2x + 10 = 22 - 3x$

$2 \cdot (-x + 5) = \frac{2 \cdot (22 -}{2}$

$\Leftrightarrow$ { Multiplicerar båda led m

$y = f(x)$



1. Funktionens graf har sin topp i punkten (☐ , ☐).
2. Högstagradstermen för polynomet som definierar funktionen har koefficienten ☐ .
3. Funktionens graf är en parabel som öppnar sig
   - ○ uppåt.
   - ○ nedåt.
4. Grafens symmetriaxel är linjen $x =$ ☐ .
5. På grund av egenskaperna i punkterna 1 och 3 kan vi säga att funktionen f
   - ○ har nollställen.
   - ○ inte har nollställen.
6. Funktionens nollställen kan lösas ur ekvationen
   - ○ $f(x) = 0$
   - ○ $f(y) = x$
   - ○ $f(0) = y$
   - ○ $f(x) = y$
   - ○ $f(-1) = y$
7. Denna ekvation är i sin normalform ☐ $x^2 +$ ☐ $x +$ ☐ $= 0$.
8. Ekvationen har därmed ☐ lösning(ar).

Workbook features

The e-book is also digital workbook that provides an integrated work environment for solving student assignments

workbook

Screenshot

Ask for teacher's comments

3.2.5: Algebra och ekvationer > Att lösa en ekvation > Uppgifter om första grader

1

Solving an assignment

$0{,}1 \cdot x - 200 = 0.2 \cdot x - 200{,}5$

$\Leftrightarrow$ { Add 200 to both sides }

$0{,}1 \cdot x = 0.2 \cdot x - 0{,}5$

Editor for structured derivation

Svar:

Editors for graphs, figures, etc

3.2.1: Algebra och ekvationer > Att lösa en ekvation > Lösningens existens

Lösningens existens

Den föregående ekvationen hade exakt en lösning. Det finns emellertid ekvationer som inte har någon lösning. Ett exempel på en sådan ekvation är $x + 2 = x$.

$x + 2 = x$

$\Leftrightarrow$ { Subtraherar x från båda leden }

$x + 2 - x = x - x$

$\Leftrightarrow$ { Förenklar båda leden }

Härledningens resultat är alltså att

$$x + 2 = x \Leftrightarrow F$$

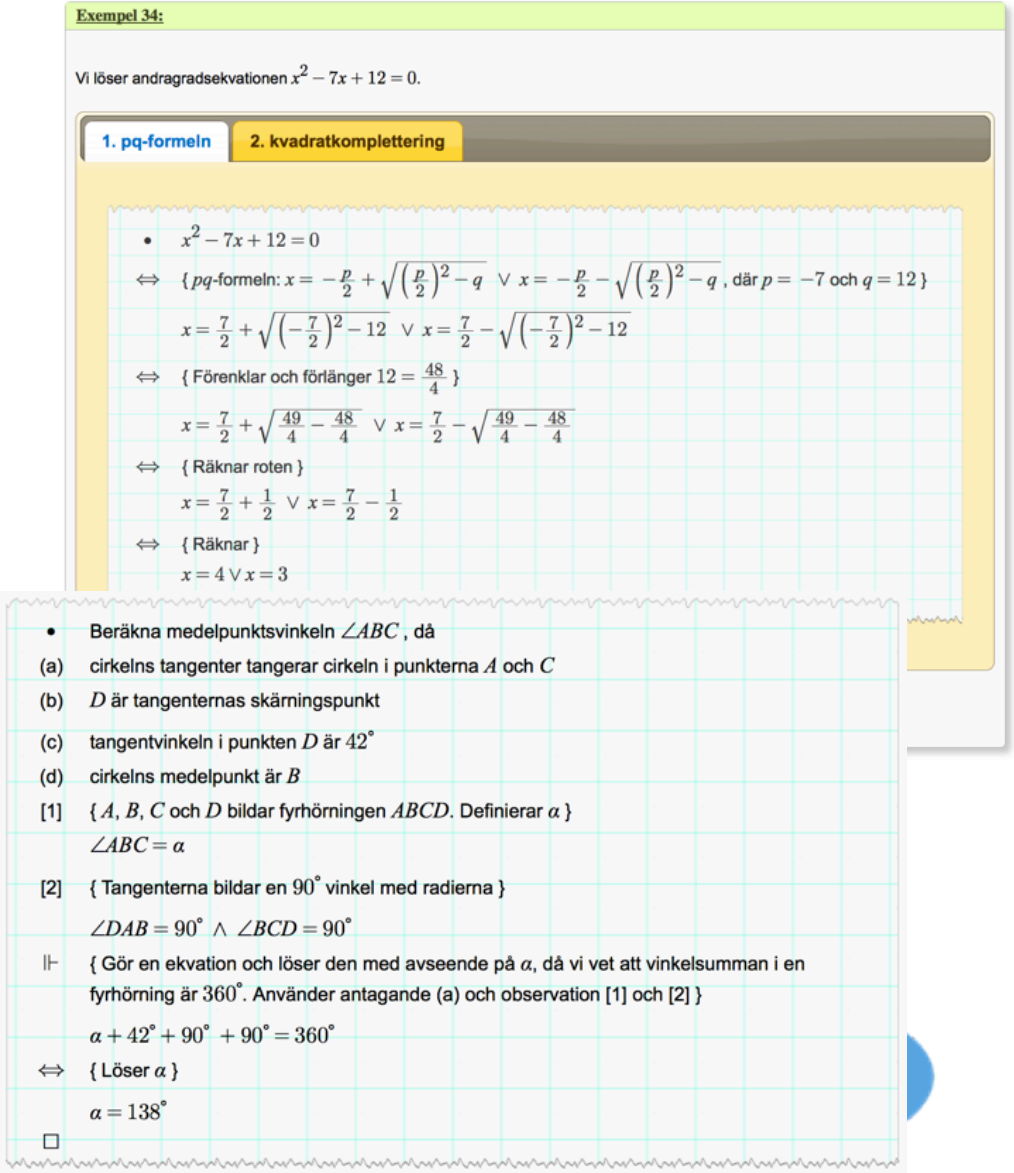
Ekvationen är falsk för alla värden på variabeln x. Därför kallar vi den för en *iden falsk ekvation*. En sådan ekvation har ingen lösning.

workbook

Editor for structured derivations

Functionality

- mathematical formulas are entered and manipulated using traditional mathematical notation
- can use shortcuts or symbol palet for entering mathematical symbols
- easy editing of structured derivations
- enforces structured derivation syntax
- can check syntactic correctness of mathematical formulas
- automatic checker for algebra course available



Exempel 34:

Vi löser andragradsekvationen $x^2 - 7x + 12 = 0$.

1. pq-formeln | 2. kvadratkomplettering

- $x^2 - 7x + 12 = 0$

$\Leftrightarrow$ { pq-formeln: $x = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 - q} \vee x = -\frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 - q}$, där $p = -7$ och $q = 12$ }

$x = \frac{7}{2} + \sqrt{\left(-\frac{7}{2}\right)^2 - 12} \vee x = \frac{7}{2} - \sqrt{\left(-\frac{7}{2}\right)^2 - 12}$

$\Leftrightarrow$ { Förenklar och förlänger $12 = \frac{48}{4}$ }

$x = \frac{7}{2} + \sqrt{\frac{49}{4} - \frac{48}{4}} \vee x = \frac{7}{2} - \sqrt{\frac{49}{4} - \frac{48}{4}}$

$\Leftrightarrow$ { Räknar roten }

$x = \frac{7}{2} + \frac{1}{2} \vee x = \frac{7}{2} - \frac{1}{2}$

$\Leftrightarrow$ { Räknar }

$x = 4 \vee x = 3$

- Beräkna medelpunktsvinkeln $\angle ABC$, då

(a) cirkelns tangenter tangerar cirkeln i punkterna A och C

(b) D är tangenternas skärningspunkt

(c) tangentvinkeln i punkten D är 42°

(d) cirkelns medelpunkt är B

[1] { A, B, C och D bildar fyrhörningen $ABCD$. Definierar α }

$\angle ABC = \alpha$

[2] { Tangenterna bildar en 90° vinkel med radierna }

$\angle DAB = 90^\circ \wedge \angle BCD = 90^\circ$

$\Vdash$ { Gör en ekvation och löser den med avseende på α, då vi vet att vinkelsumman i en fyrhörning är 360°. Använder antagande (a) och observation [1] och [2] }

$\alpha + 42^\circ + 90^\circ + 90^\circ = 360^\circ$

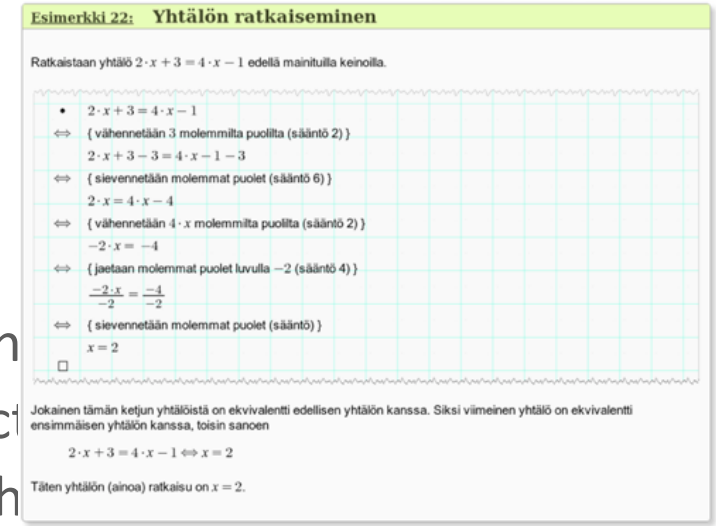
$\Leftrightarrow$ { Löser α }

$\alpha = 138^\circ$

$\square$

workbook

Automatic checking of derivations



Esimerkki 22: Yhtälön ratkaiseminen

Ratkaistaan yhtälö $2 \cdot x + 3 = 4 \cdot x - 1$ edellä mainituilla keinoilla.

- $2 \cdot x + 3 = 4 \cdot x - 1$

$\Leftrightarrow$ { vähennetään 3 molemmilta puolilta (sääntö 2) }

$2 \cdot x + 3 - 3 = 4 \cdot x - 1 - 3$

$\Leftrightarrow$ { sievennetään molemmat puolet (sääntö 6) }

$2 \cdot x = 4 \cdot x - 4$

$\Leftrightarrow$ { vähennetään $4 \cdot x$ molemmilta puolilta (sääntö 2) }

$-2 \cdot x = -4$

$\Leftrightarrow$ { jaetaan molemmat puolet luvulla -2 (sääntö 4) }

$\frac{-2 \cdot x}{-2} = \frac{-4}{-2}$

$\Leftrightarrow$ { sievennetään molemmat puolet (sääntö) }

$x = 2$

$\square$

Jokainen tämän ketjun yhtälöistä on ekvivalentti edellisen yhtälön kanssa. Siksi viimeinen yhtälö on ekvivalentti ensimmäisen yhtälön kanssa, toisin sanoen

$$2 \cdot x + 3 = 4 \cdot x - 1 \Leftrightarrow x = 2$$

Täten yhtälön (ainoa) ratkaisu on $x = 2$.

- Computer checks each step in a student's derivation
  - analyzes the logic behind student's solution
  - marks the steps that it cannot prove correct
  - logic of solution is checked, not only wheth answer is correct
- Student gets immediate feedback on the correctness of own solution
- Requires server support:
  - need for powerful automatic theorem pro s and CAS systems
  - software for interpreting students' solutions and providing feedback

workbook

Automatic checking of derivations

Student submits solution for computer checking

Student's solution

•	Simplify $\cos(x+\frac{\pi}{3})$, when
-	$\sin x = \cos x$
$\Vdash$	{transitivity}
	$\cos(x+\frac{\pi}{3})$
$=$	{cosine for sums: $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$}
	$\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}$
$=$	{substitute values $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$}
	$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$
$=$	{assumption}
	$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\cos x$
$=$	{factor out $\cos x$}
	$\frac{1-\sqrt{3}}{2}\cos x$
$\square$	$\cos(x+\frac{\pi}{3}) = \frac{1-\sqrt{3}}{2}\sin x$

workbook

Automatic checking of derivations

Student submits solution for computer checking

Computer marks those derivation steps that it cannot prove correct

Student's solution

•	Simplify $\cos(x+\frac{\pi}{3})$, when
-	$\sin x = \cos x$
$\Vdash$	{transitivity}
	$\cos(x+\frac{\pi}{3})$
=	{cosine for sums: $\cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$}
	$\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}$
=	{substitute values $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{3} = \frac{1}{2}$} ?
	$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$
=	{assumption}
	$\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\cos x$
=	{factor out $\cos x$}
	$\frac{1-\sqrt{3}}{2}\cos x$
□	$\cos(x+\frac{\pi}{3}) = \frac{1-\sqrt{3}}{2}\sin x$?

workbook

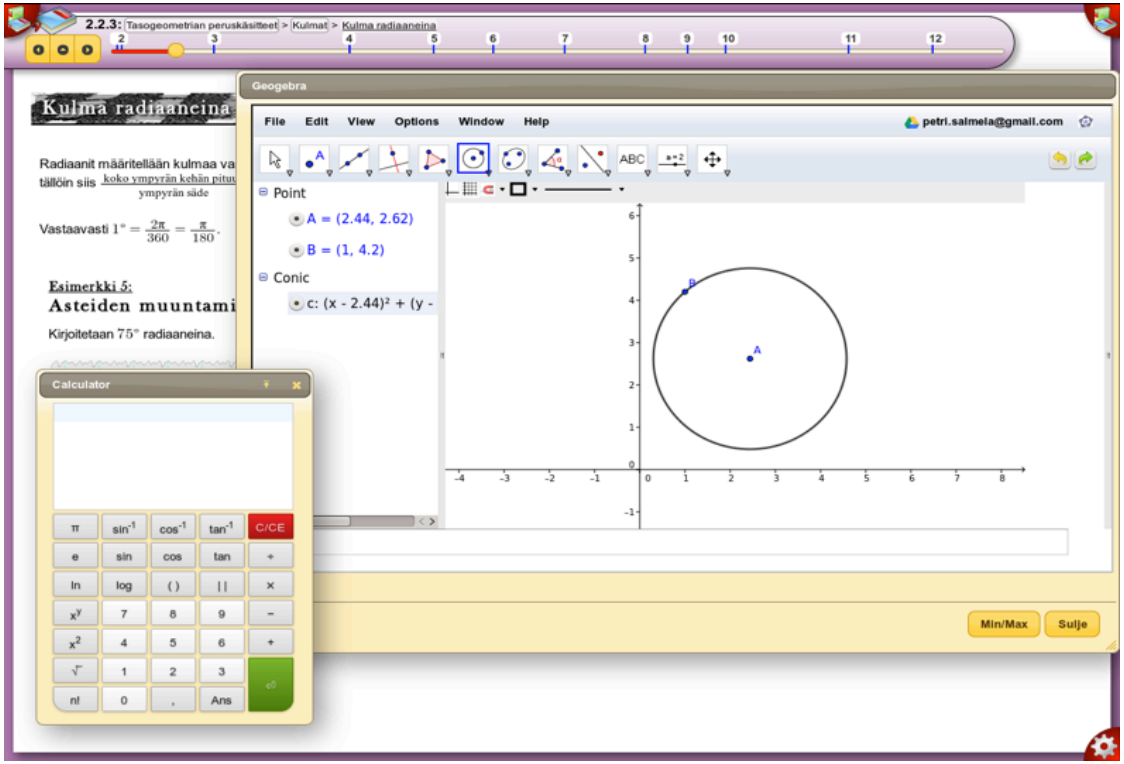
Special tools

Special editors for

- drawing geometric figures
- function graphs
- tables
- free text with mathematical symbols

Standard scientific calculator available

Can also use web based tools directly in the e-book



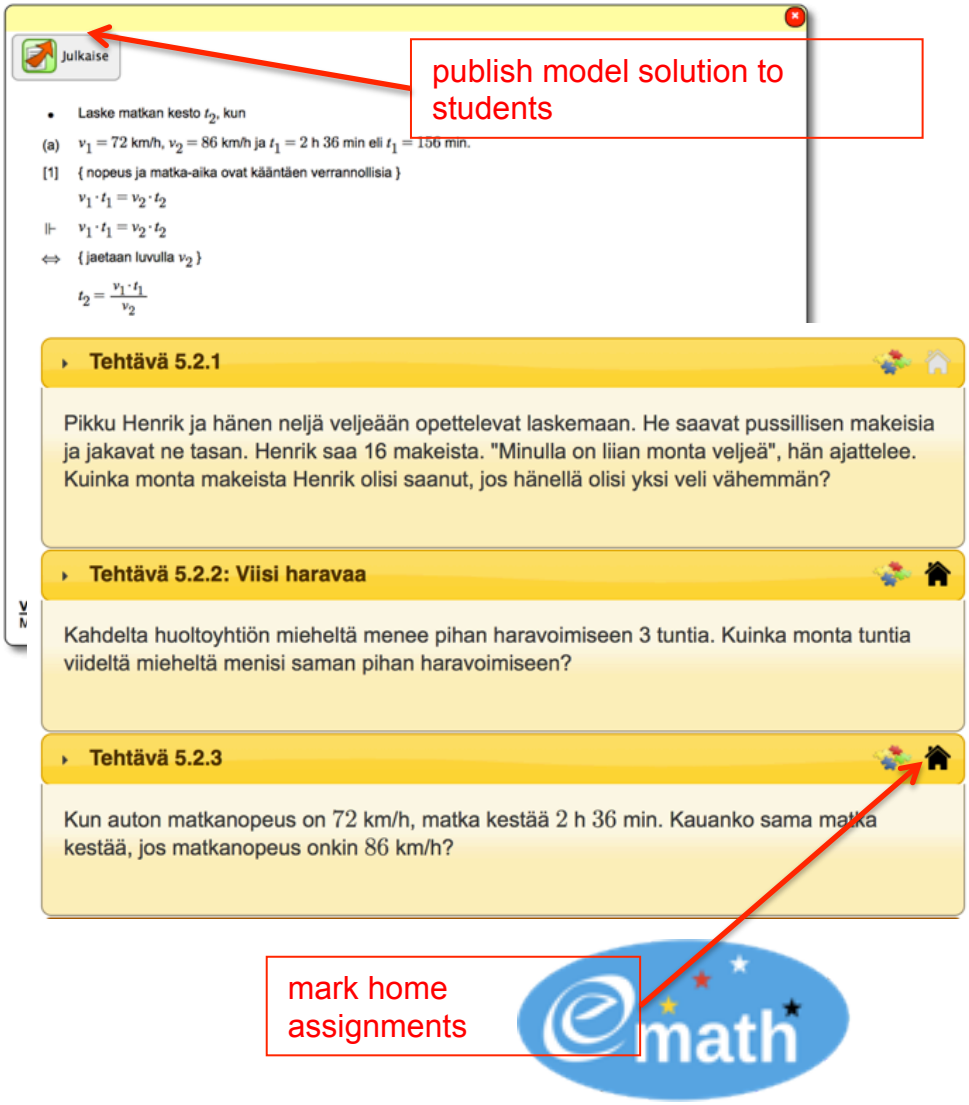
Classbook features

The e-book handles communication between teacher and students

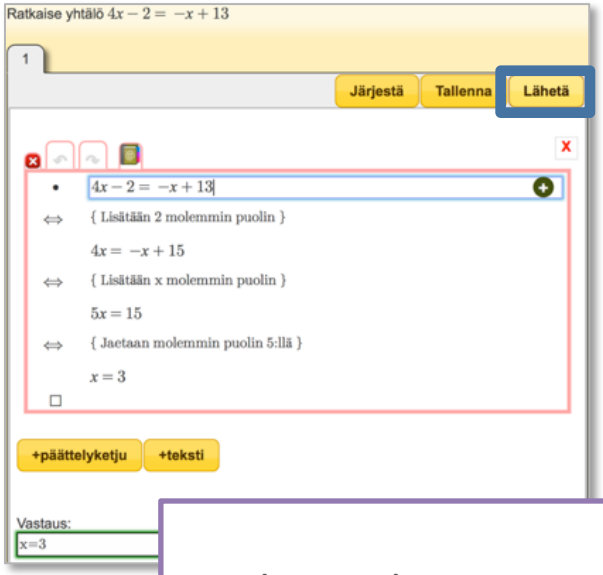
classbook

Add content to students e-books

- Teachers can publish own material directly in students' books, e.g.
  - new theory, examples, assignments
  - marking home assignments
  - give model solutions to selected assignments
- New material can be added to existing textbook pages



classbook



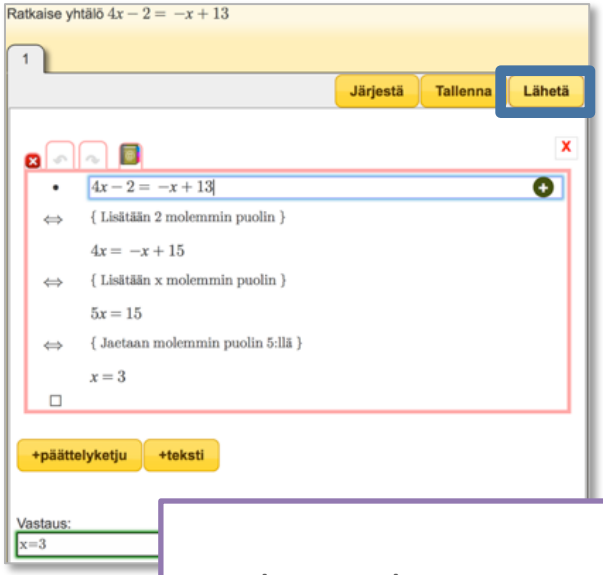
Student submits own solution to teacher by the click of a button

Personal feedback on student's solution

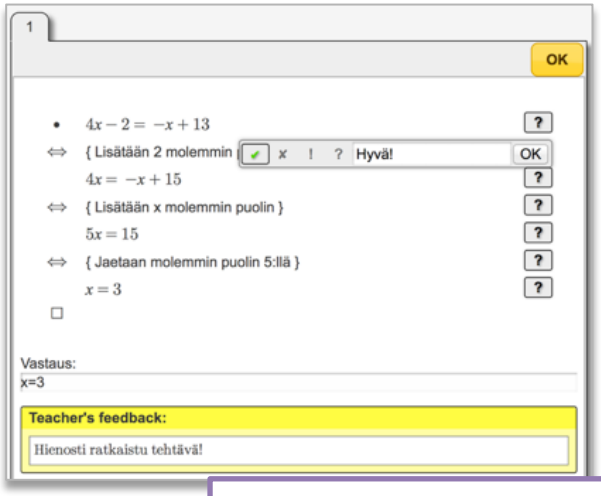
Student

classbook

Personal feedback on student's solution



Student submits own solution to teacher by the click of a button



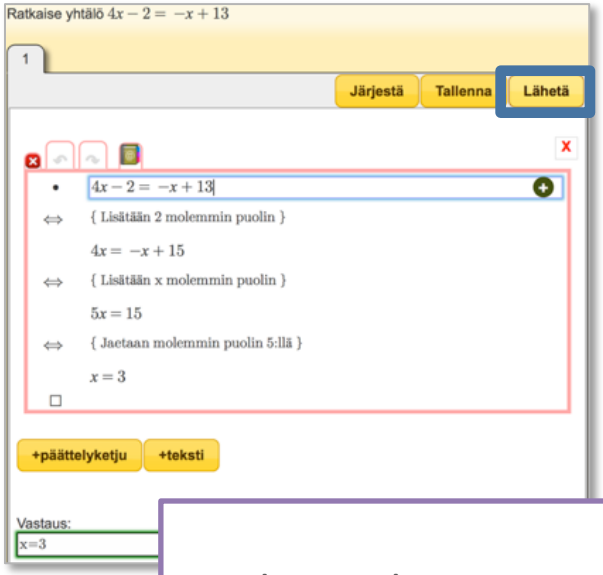
Teacher reviews the solution in own book and comments on students solution

Student

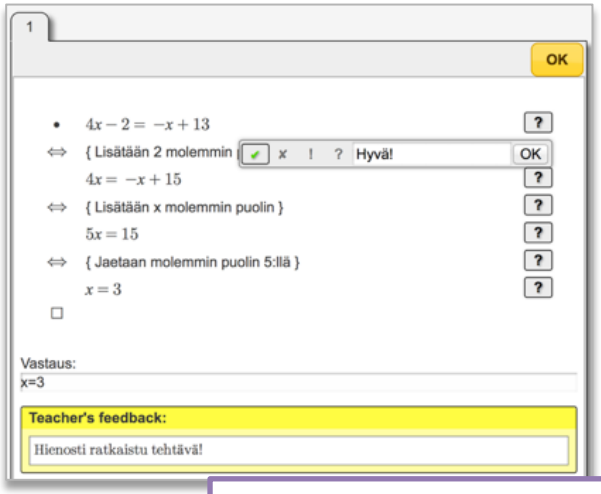
Teacher

classbook

Personal feedback on student's solution

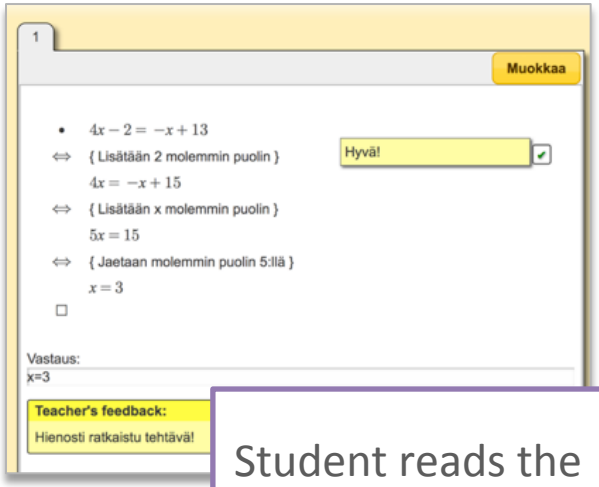


Student submits own solution to teacher by the click of a button



Teacher reviews the solution in own book and comments on students solution

Student



Student reads the feedback in own book

Student

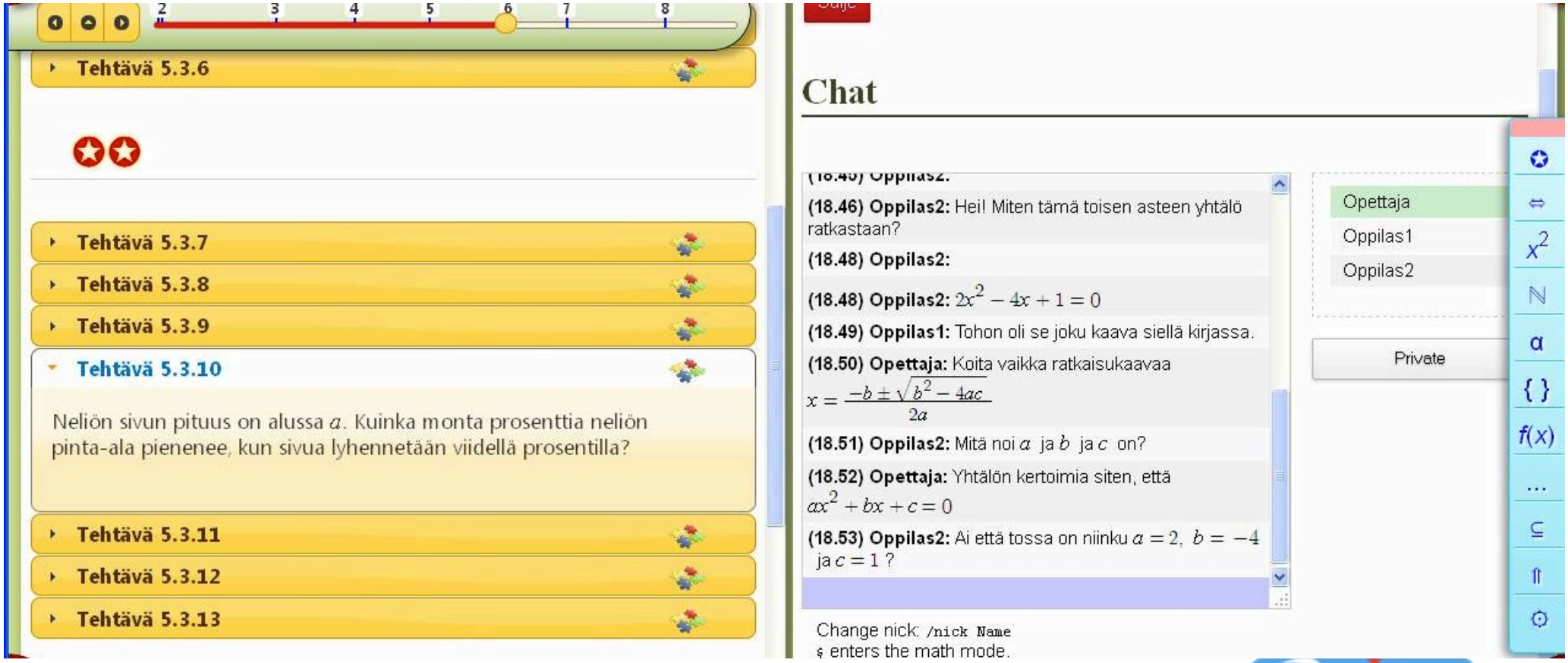
Teacher

classbook

Class net

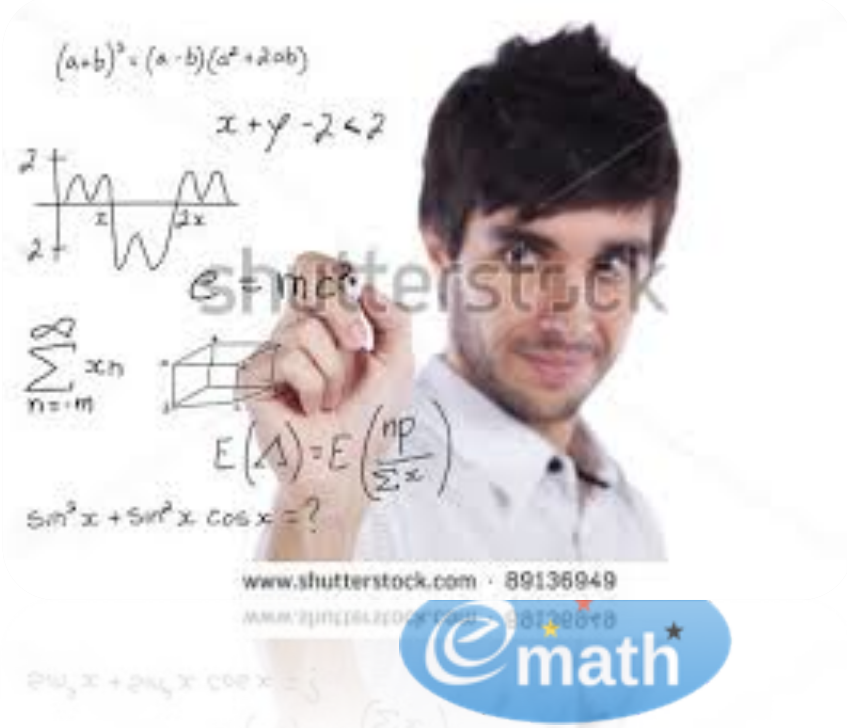
- chat room

can write mathematical formulas in chat entries



Teacherbook features

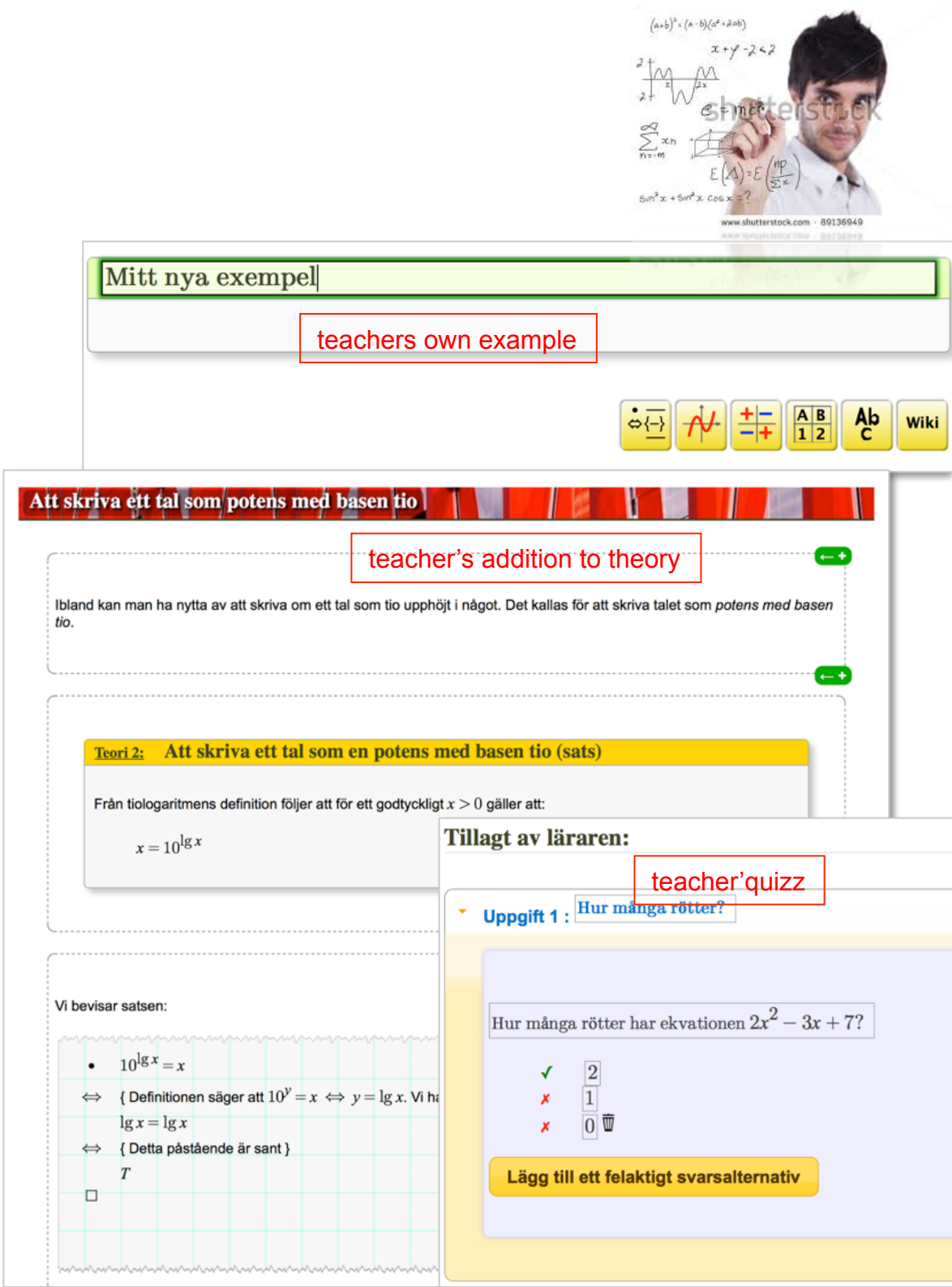
The e-book allows teacher to create own course material



teacherbook

Add own material

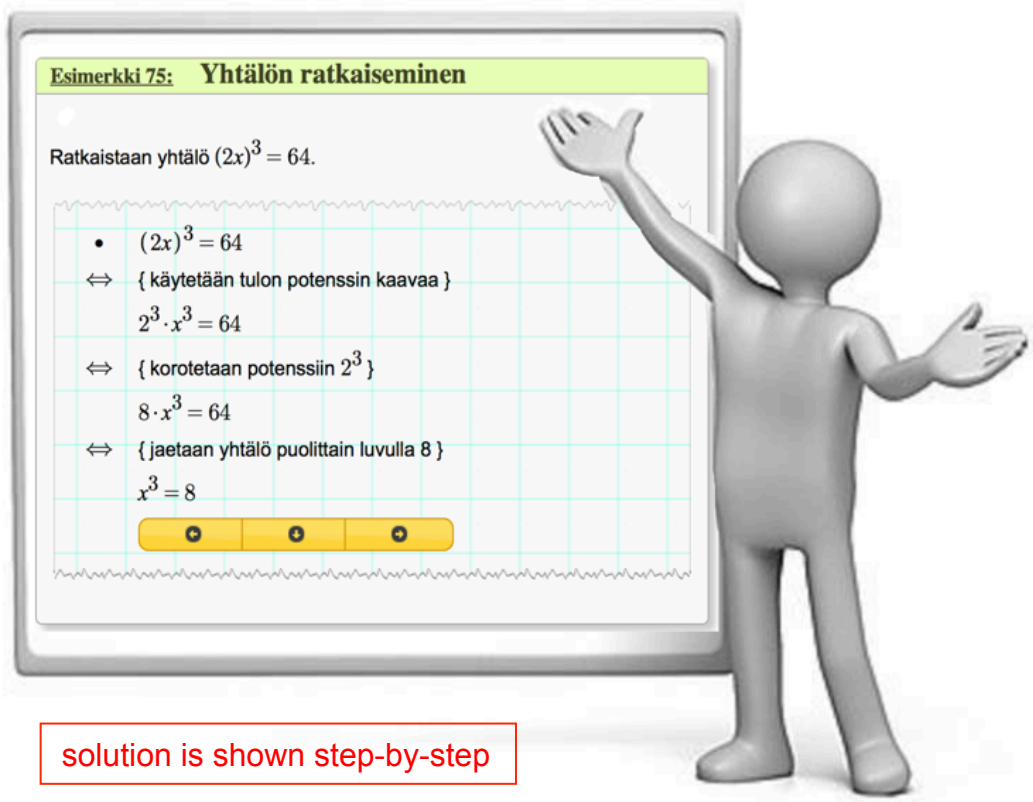
- Teacher creates own teaching material in ebook using special editors:
- add new theory, examples,
- add new quizzes and assignments
- add pictures, figures, videos, sound, etc
- can add more or less any material that can be presented on web pages



teacherbook

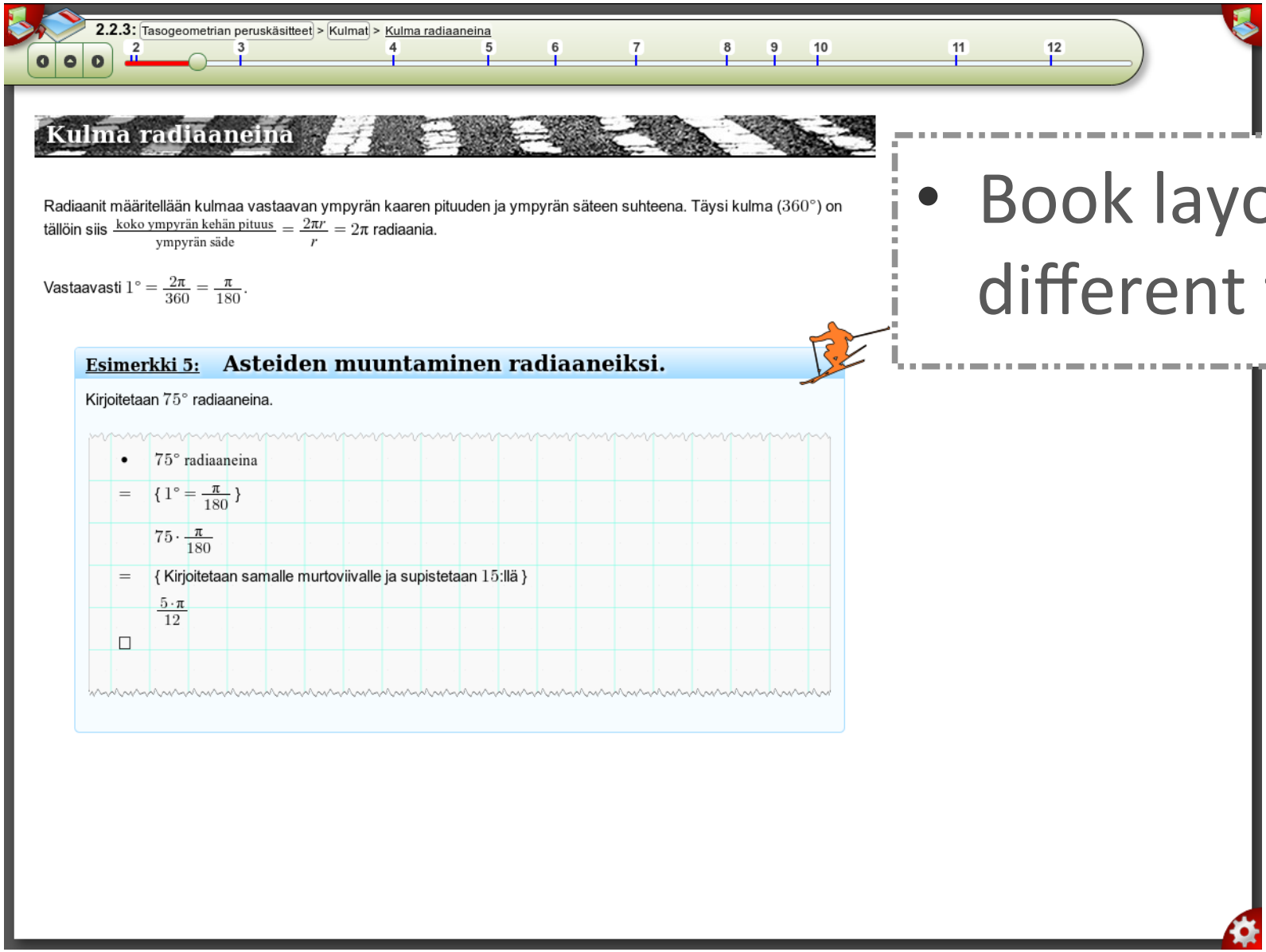
Presentation tools

- Teachers can create PowerPoint-like presentations from eMath textbook, workbook, and own material
- Structured derivations can be shown step-by-step, making them easy to explain and follow
- Interactive onscreen manipulation of graphs, geometric figures, tables
- Possibility to add own pictures, videos, sounds, etc.



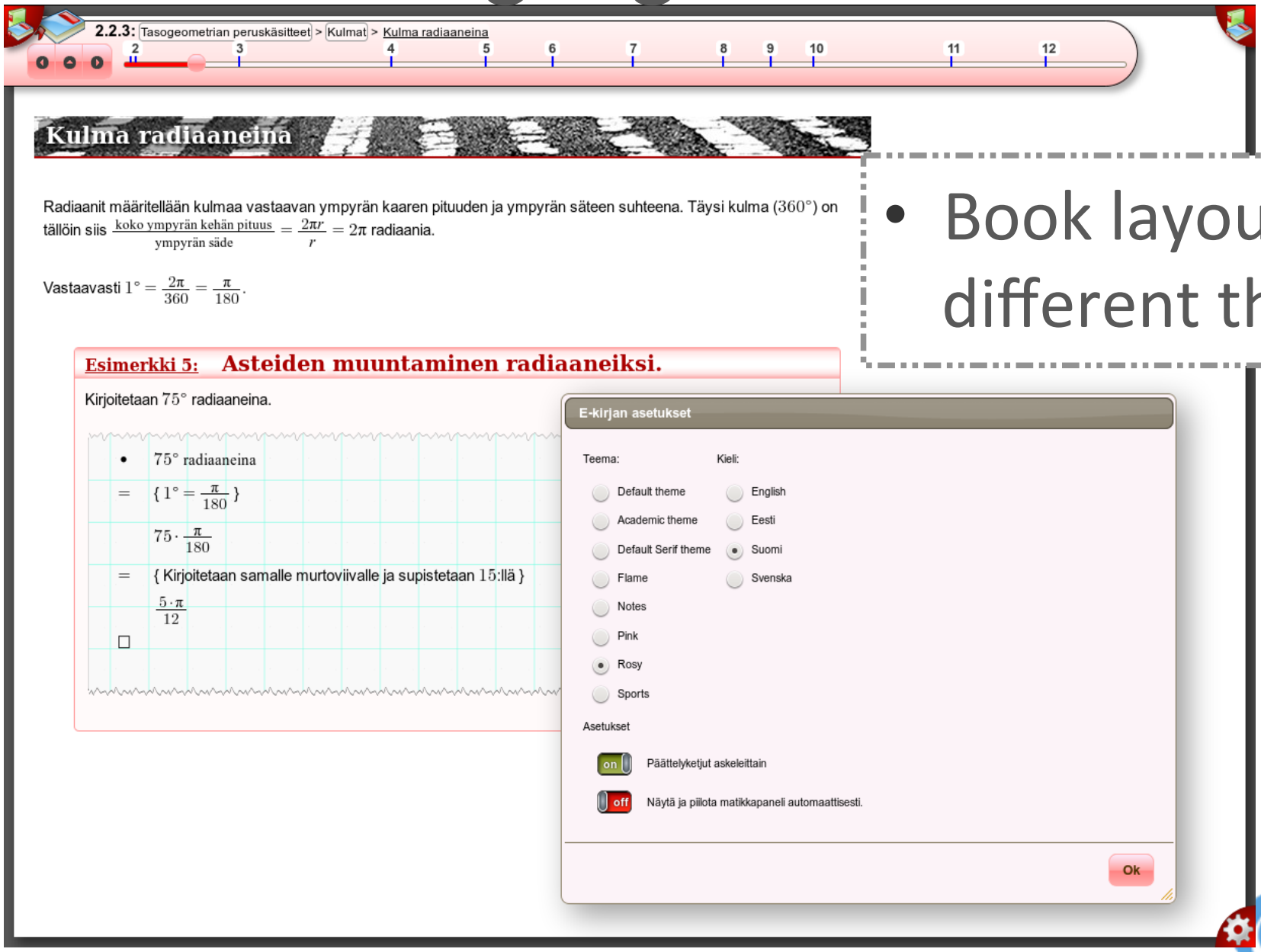
solution is shown step-by-step

Language and themes



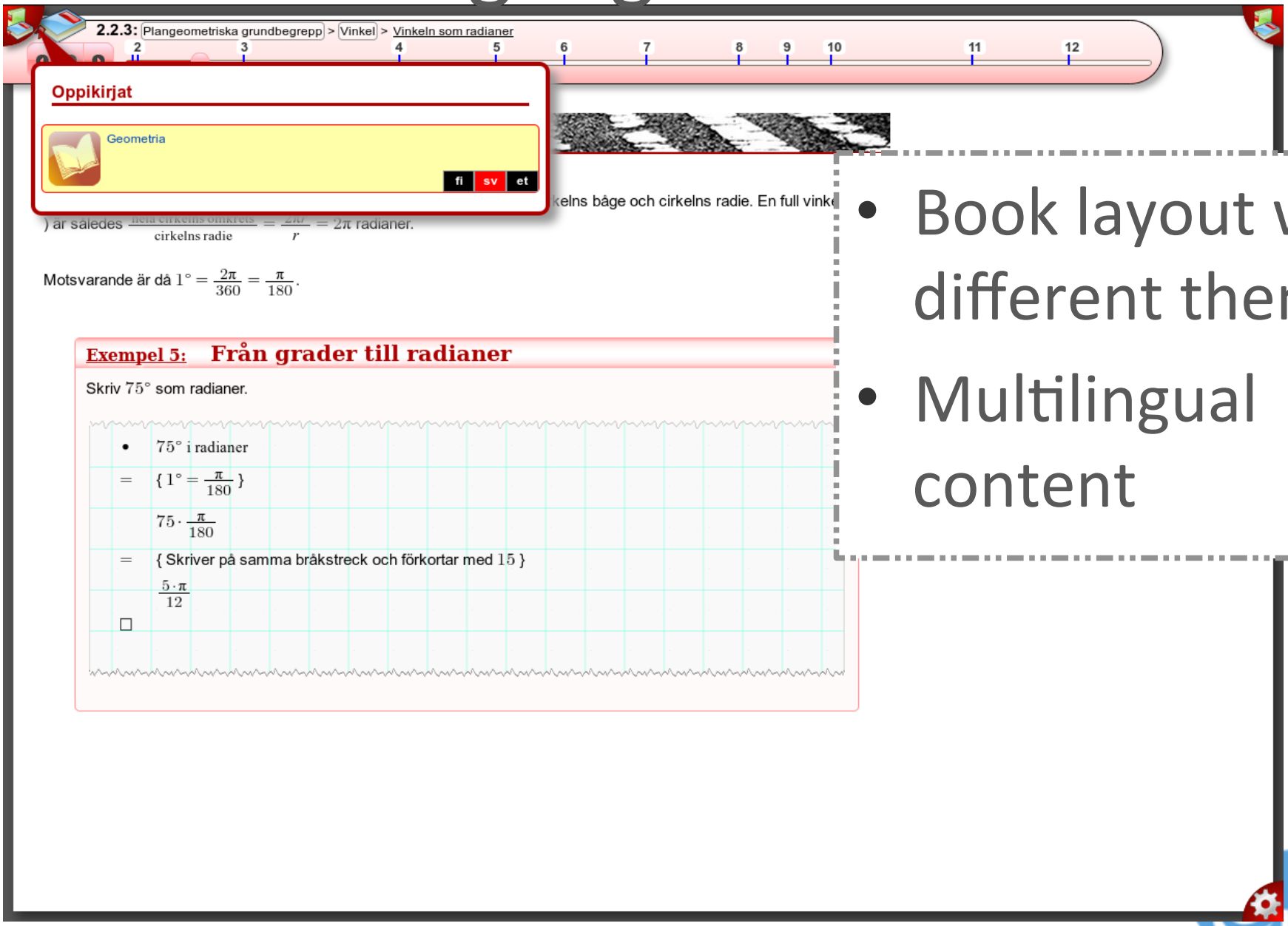
- Book layout with different themes

Language and themes



- Book layout with different themes

Language and themes



- Book layout with different themes
- Multilingual content

How the E-math project continues

- Project 1: write e-books for all high school courses in advanced math 2014 – 16 in Finland
- Project 2: create a virtual high school in Finland based on E-math technology 2014 – 16
- Startup company: Four Ferries commercializes the E-math technology
  - IPR transfere agreement with Åbo Akademi University and University of Turku
  - project members main shareholders in company

Project 1: E-books for high school math

- Create e-books based on structured derivations for full mathematics curriculum in high school (advanced level)
  - E-math project built 4 e-books
  - New project will add 7 more e-books
- Project funded by Swedish cultural foundation in Finland and Finnish technology industry foundation
- E-books for all compulsory courses in high school math (MAA 1 – MAA 10)
  - All books follows Finnish curriculum requirements
  - All books bilingual, Finnish and Swedish (language selected in e-book)
  - All books will be freely downloadable
  - All books completed in 2016
- Teaching using these e-books has already started in pilot schools this autumn

Pilots autumn 2014

Swedish language high schools in Finland

Vasa gymnasium
Vasa svenska aftonläroverk
Vasa övningsskola
Närpes gymnasium
Björneborgs svenska samskola
Korsholms högstadium
Kristinestads högstadium
Gymnasiet svenska normallyceum
Tölö specialiseringsgymnasium
Ekenäs gymnasium
Ekenäs högstadieskola
Borgå Gymnasium
Kotka svenska samskola
Sibbo gymnasium

Finnish language high schools in Finland

Raision lukio
Luostarivuoren lukio
Kerttulin lukio

Sweden

Mediagymnasiet, Nacka

Project 2: Virtual high school project

- Project funded by Swedish cultural foundation in Finland
- E-book works also for virtual courses
  - does not require that students and teachers are in the same room,
  - all communication happens over the net
- E-book will be used for different kinds of school subjects
  - presently only math is supported,
  - but we are building support for other topics
- Example books being built or under consideration
  - social studies
  - programming methods
  - philosophy
  - advanced physics