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# Using dynamic-geometry and computer-algebra systems in problem-based courses for future engineers

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## Abstract

It is a modern trend today when formulating the curriculum of a geometric course at the technical universities to start from a real-life problem originated in technical praxis and subsequently to define which geometric theories and which skills are necessary for its solving. Nowadays, interactive and dynamic geometry software plays a more and more important role in this scenario as it helps to think algorithmically, enables to discuss the solvability of the whole class of geometric problems from different point of views and mainly serves as a first step to variational geometry needed later in geometric modelling. This makes the teaching and learning process more efficient and also more interesting for students. In our contribution, we present and discuss some particular examples of this approach.

## 1 Motivation

Nowadays, when teachers or lecturers at technical universities formulate (or innovate) curricula of geometric courses, they use more and more often the so-called problem-based teaching approach. It reflects a need of students to know why this theory from mathematics or geometry is taught and required. So, it is especially useful and efficient to start from some existing real-life problem originated in technical praxis and consequently to explore which theories and which skills are necessary for solving it. Without any doubts, it is important to motivate students by real-life problems from their field of study. Our experience shows that interactive and dynamic geometry may play a significant role in this scenario since it helps students to think algorithmically, enables to discuss the solvability of the whole class of geometric problems and mainly serves also as a first step also to geometric modelling and Computer-Aided Design (CAD).

In our contribution we will present this approach on one particular example taken from the geometry course for mechanical engineering. Especially, we will emphasize the application of dynamic geometry software (DGS) supported by a suitable computer algebra system (CAS). Our teaching

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approach is characterized by a distinguished structure. First, we choose a suitable real engineering problem. Then we subdivide it to a number of elementary sub-problems. And finally we discuss how the curriculum and the corresponding teaching procedures shall be chosen to satisfy the goal to teach the students to solve the selected real-life problem (used at the beginning for motivation).

We will present our experience with the problem-based teaching of geometry using dynamic geometry software and discuss some aspects of this approach. Of course, when modifying the curricula we cooperate closely with the colleagues from the Faculty of Mechanical Engineering and discuss all the real-life problems with the specialists. The discussion with the PhD students and their mentors are especially useful and it already led to many innovations of curricula.

In addition, we would like to stress out that dynamic geometry software supported by the computer algebra systems is a very powerful tool for making teaching and learning process more effective and more efficient. Sadly, a majority of computer-aided learning and teaching activities for geometry courses is still oriented only on fundamental problems of elementary and secondary schools and one can find a very few scenarios prepared for university students (see [14], [8], [7], [4]). Nonetheless in all our contributions, we strive to persuade the readers who are interested in this topic that DGS is a modern geometric teaching tool suitable also for the technical universities (see [12]).

## 2 Real-life problems for teaching purposes

In what follows, we plan to present our experience with application of modern geometric teaching tools at the university. We will stress out a didactic role of these systems (DGS and CAS) for solving not only simple planar questions but also for more complicated spatial constructions originated in practical engineering problems, i.e., dealing with curves and surfaces from special classes and derived objects.

We start with presenting a particular real-life engineering problem accessible also for students (namely, modelling solid twist drill flute), then we subdivide it to some partial sub-problems that will be solved separately. In Section 4, we show how the curriculum and the corresponding teaching procedures are chosen to satisfy the goal to teach the students to solve this chosen real technical problem.

### 2.1 Modelling solid twist drill flute

*Solid twist drills* (see Fig. 1) are produced by the technology of grinding (see Fig. 2). During the manufacturing process the grinding wheel makes the final shape of the cutting tool (the twist drill flute). The model of the solid twist drill is usually created using some CAD system (for instance CATIA V5). However, the standard CAD systems do not support the operation of dragging a solid body of the grinding wheel along the desired curve; which is necessary for the good correlation between the model and the real prototype. Hence, it is important to construct a curve which is the generator of a helical surface, i.e., the flute of the solid twist drill.

We recall that *helical surfaces* are generated by screw motion of a generating curve (*generatrix*). The screw motion is a composition of a revolution about a given axis and a translation along this axis, i.e, it is determined by the *axis*, the *parameter* and the *orientation* (+/−). Trajectories of the points of

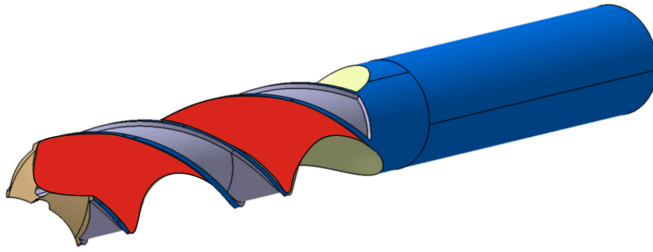


Figure 1: Twist drill flute (red)

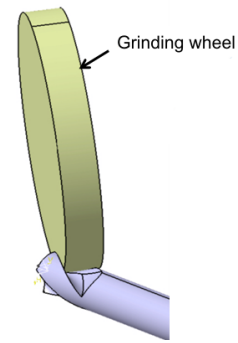


Figure 2: Grinding wheel (yellow)

the generatrix are curves called *helices*. The helical surfaces are very often used in the technical practise, to the above mentioned case we add e.g. helical gears, spiral staircases or heliclines.

The technique how to construct the generator curve of the helical surface is a method for the construction of the characteristic of envelope surfaces. Hence for further right understanding of the topic, we have to also recall another class of surfaces – the so called envelope surfaces. Consider a one-parameter set  $\sigma(t)$  of surfaces in 3D-space. The *envelope surface* of this system is the surface which is tangent to every surface from  $\sigma(t)$  along a curve  $c(t)$ . This curve is called the *characteristic curve* of the system. Obviously, the envelope surface can be generated by the motion of any of its characteristics.

So, the problem of determining the envelope surface can be reduced to the problem of constructing its characteristic curve. For our purposes, we present the method of inscribe spheres which is often used for finding the characteristics. This method is suitable for surfaces of revolution and it is based on two fundamental observations. First, it is easy to inscribe a sphere into an arbitrary surface of revolution, and secondly, we know how to construct the characteristic of a moving sphere (it is the circle lying in the plane perpendicular to the direction of the motion, midpoint of which is the same as for the sphere and whose radius equals the sphere radius). Firstly, we construct (or compute) an auxiliary sphere  $\kappa$  which touches the moving surface of revolution  $\Phi$  along the circle  $k$ . Then we construct (or compute) the characteristic curve  $c$  of the sphere  $\kappa$  and find the intersection points of the circles  $k$  and  $c$ . We have to repeat this construction several times to obtain enough points of the characteristic curve.

## 2.2 Modelling solid twist drill cutting edge

*The cutting edge* or *the lip* of the two-flute drill extends from the chisel edge to the outer edge. In particular, it is the intersection of *the twist drill flute* (the yellow surface in Fig. 3) and *the major flank face* (the green surface in Fig. 3). A goal of mechanical engineers is to test the influence of changing some chosen (known) parameters on the curvature of the cutting edge.

We divide this problem into a suitable number of sub-problems which will be later reflected in different teaching topics in the curricula. The identified sub-problems are:

- Parameterizing the surface of the twist drill flute;

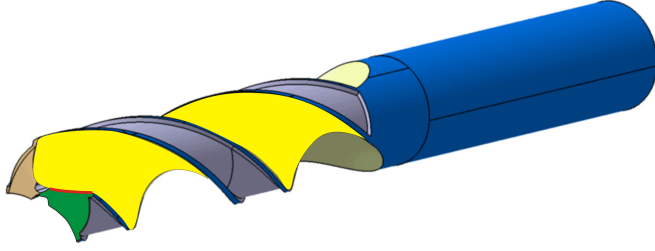


Figure 3: The cutting edge (red) is the intersection of the twist drill flute (yellow) and the major flank face (green)

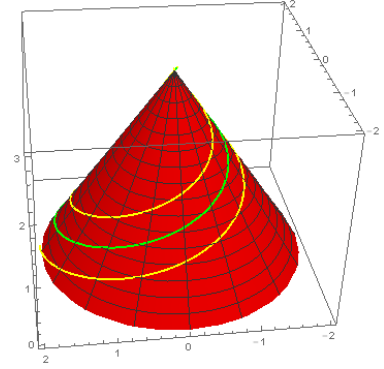


Figure 4: Spiral curve on a cone

- Finding the equation of the major flank face;
- Computing the intersection of these two surfaces;
- Solving the global problem.

Firstly, we have to parameterize the helical surface. Clearly, when we apply the smooth helical motion to the spatial curve  $c$ , we obtain the helical surface. In this case, the profile curve  $c$  consists of the three segments (see Fig. 5), i.e., parts of the two circles and a segment of the straight line. Both circles are given by their centers  $(S_1, S_2)$  and their radii  $(R_1, R_2)$ . The straight line is given by the points  $K, L$ . Thus, the parameterizations of the parts (in the homogeneous coordinates with the fourth coordinate as the homogenizing part) are given as

- $line = (L[1] + v(K[1] - L[1]), L[2] + v(K[2] - L[2]), L[3] + v(K[3] - L[3]), 1)$
- $circle1 = (R_1 \cos v + S_1[1], R_1 \sin v + S_1[2], 0, 1)$
- $circle2 = (R_2 \cos v + S_2[1], R_2 \sin v + S_2[2], 0, 1)$

In this real case, the situation is more complicated as the trajectory of the curve  $c$  during the motion is a spiral curve – it does not lie on a cylinder (as in the ideal case) but it lies on a cone of revolution (see [11] and Fig. 4). We recall that the spiral motion is given by the matrix

$$\begin{pmatrix} \cos(u) \left( \frac{2uv_0(\frac{D_1}{2} - \frac{D_2}{2})}{D_2 v_k} + 1 \right) & \sin(u) \left( \frac{2uv_0(\frac{D_1}{2} - \frac{D_2}{2})}{D_2 v_k} + 1 \right) & 0 & 0 \\ \sin(u) \left( -\frac{2uv_0(\frac{D_1}{2} - \frac{D_2}{2})}{D_2 v_k} - 1 \right) & \cos(u) \left( \frac{2uv_0(\frac{D_1}{2} - \frac{D_2}{2})}{D_2 v_k} + 1 \right) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & uv_0 & 1 \end{pmatrix}, \quad (1)$$

where  $v_0$  is the parameter of the spiral motion,  $D_1, D_2$  are the radii of the bases of the truncated cone and  $v_k$  is its height.

The parameterizations of the three parts of the helical surface (solid twist drill flute) are easily obtained using the matrix multiplication of the parameterizations of the parts of the curve  $c$  (written in

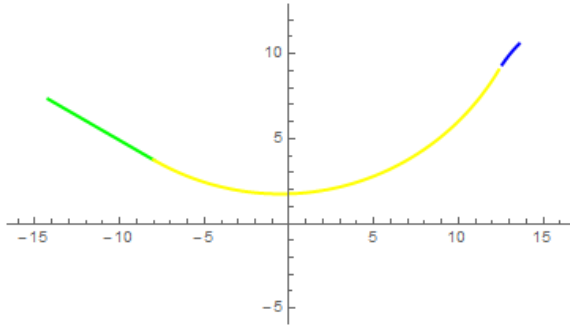


Figure 5: The three segments of the curve  $c$  (the segment of the straight line is green, the first circle yellow and the second circle blue)

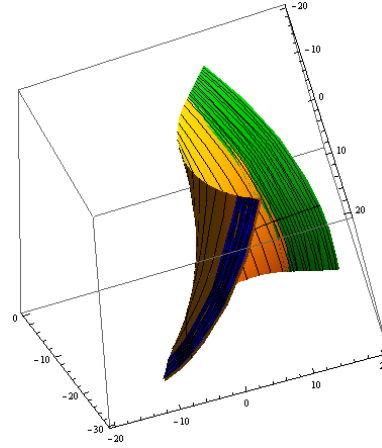


Figure 6: The helical surface obtained from the generatrix  $c$

the row vector form) and the matrix of the helical motion. A part of the resulting surface is illustrated in Fig. 6.

Now, we are supposed to parameterize the part of the drill which is called the point and it consists of parts of two cones (one of them is the green surface in Fig. 3). The shape of these two cones are identical, but they are only symmetrically located with respect to the axis of the drill. Hence it is enough to compute only one of them. The cone is given by the axis and the generatrix of the cone. We compute number  $a = \tan \alpha$ , where  $\alpha$  is the angle between axis and generatrix.

We continue with the matrix representation of the cone in the canonical position (the axis of this cone is the  $z$ -axis and the vertex is the origin)  $X \cdot Q \cdot X^T = 0$ , where  $X = (x, y, z, 1)$  and

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (2)$$

To move the cone in the canonical position to the right position we have to use a suitable transformation. In this situation this is the composition of the two rotations ( $R_y, R_y$ ) and one translation  $T$ . The next three matrices describe these mentioned transformations:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_1 & p_2 & p_3 & 1 \end{pmatrix}, \quad (3)$$

$$R_y = \begin{pmatrix} \cos \beta & 0 & -\sin \beta & 0 \\ 0 & 1 & 0 & 0 \\ \sin \beta & 0 & \cos \beta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

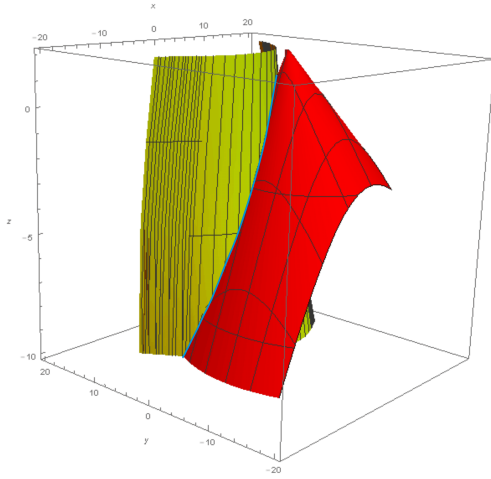


Figure 7: The intersection of the helical surface (green) and the the cone (red)

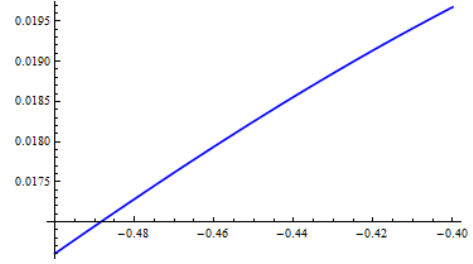


Figure 8: The curvature of the intersection curve (the cutting edge)

$$R_z = \begin{pmatrix} \cos \gamma & \sin \gamma & 0 & 0 \\ -\sin \gamma & \cos \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (5)$$

So, the matrix describing the cone as a quadric surface is obtained in the form

$$A = T \cdot R_z \cdot R_y \cdot Q \cdot R_y^T \cdot R_z^T \cdot T^T \quad (6)$$

and its equation is

$$X \cdot A \cdot X^T = 0, \quad (7)$$

where  $X = (x, y, z, 1)$ .

Finally, we want to compute the intersection of the helical surface and the cone. For this purpose, one can use e.g. CAS Wolfram Mathematica. Anyway, the result of this computation is too complicated and does not give us a right view of the geometrical situation. From this reason we reparameterize the helical surface, in particular the two given generating circles. We use the NURBS representation (instead of the previous trigonometric one), see [10], and repeat the whole procedure once again. As one can see, this approach is more efficient and we arrive at the representation of intersection curve (see Fig. 7). Furthermore, we can easily compute the curvature of the intersection curve (see Fig. 8).

### 3 Computer-supported teaching courses

In the previous section we discussed a real-life problem which helps us to choose particular topics to the curriculum of the geometric course for mechanical engineering. Now we want to discuss a suitable tools which can support the teaching process. The requirement is clear – these tools must be

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close to the future profession of the students, i.e., the students must see a close connection to the tools of the mechanical engineering, especially to the Computer-Aided Design.

We recall, that mechanical engineering is the branch of engineering applying the principles of mathematics, physics and materials science for the design, analysis, manufacturing, and maintenance of mechanical systems. It involves the design, production, and operation of machinery and tools. Nowadays, mechanical engineers intensively use for their work Computer-aided design (CAD) systems as e.g. CATIA. It increases the productivity of the designers, improves the quality of design, helps to the communications through documentation and creates a database for manufacturing.

Our experience shows that dynamic geometry software GeoGebra and 3D modeler Rhinoceros along with the computer algebra system Wolfram Mathematica (or Wolfram Alpha) are especially suitable tools for the first steps from the descriptive geometry, projective geometry and the geometry of curves and surfaces towards geometric modelling and hence also towards CAD and CAM (computer-aided manufacturing). This software brings new opportunities and challenges how to make the teaching and learning process at technical universities more effective and simultaneously more interesting. And in addition, it significantly influences the curricula (see the next section). In this section we will shortly present this software which can be efficiently used for computer-aided geometry teaching. All three tools have been used at our university for many years and one can find a very good evaluation done by students of all courses which were modernized using this software.

GeoGebra is dynamic mathematics software which offers geometry, algebra and calculus tools in a user-friendly environment (see [1], [6]). Currently, GeoGebra represents one of the most popular and innovative open-code mathematical software working on several operating system platforms, which one can freely download from the website [www.geogebra.org](http://www.geogebra.org) (see [2]). From the geometric point of view, GeoGebra is dynamic (or interactive) geometry software that supports constructions with points, lines, conic sections and point sets. The fundamental feature of GeoGebra interface is to provide two presentations of each mathematical object in the algebra and graphics window. Teachers and students can simultaneously use a computer algebra system and an interactive geometric system. If one changes an object in one of these windows (e.g. moves points, lines etc.) its presentation in the other one will be immediately also updated (e.g. coordinates or equations are newly recomputed).

However, GeoGebra is very useful and sophisticated software especially for planar constructions. Clearly, when we prepare courses on descriptive geometry we also need to show different situations and animations in 3D. However, 3D GeoGebra, which is now being developed, is still not such a piece of software which can help to visualize problems e.g. related to surfaces. Hence, we simultaneously use software Rhinoceros or shortly Rhino for visualization of 3D shapes, see <http://www.rhino3d.com>. Rhino is a 3D NURBS (Non-Uniform Rational B-splines) curve, surface, and solid modeller which offers to users to create models of (almost) any shape (see [5]). In addition, it is often used by graphical and industrial designers and thus students can get used to software which is not only didactic but solves also technical problems from real practice. Rhino can create, edit, analyze, document, render, animate, and translate NURBS curves, surfaces, and solids with no limits on complexity, degree, or size. Rhino also supports polygon meshes and point clouds. It offers fast 3D graphics, unlimited viewports, shaded working views, perspective working views, and coordinate read out. On the other hand, one of disadvantages of Rhino software could be that it is not (compared to e.g. Geogebra) dynamic. Therefore, it is very efficient to combine advantages of both software.

Another piece of software suitable for teaching purposes reflecting the real-life problems originated in technical practise is computational software program Mathematica (see [15], [13]). It is a

computer algebra system suitable for numerical and symbolic computations. It is used in many scientific, engineering, mathematical and computing fields, based on symbolic mathematics. Mathematica can be used from teaching simple concepts in the classroom to doing serious research. It helps students to solve complicated mathematical problems which would stay unsolved otherwise. Features of Mathematica include e.g. elementary and special mathematical function libraries, matrix and data manipulation tools, solvers for systems of equations, numeric and symbolic tools for discrete and continuous calculus, computational geometry in 2D and 3D, finite element analysis and many other functions.

## 4 Course curriculum based on a real-life problem

The following diagram (see Fig. 9) shows how the real engineering problem, presented and discussed in the previous section, can be suitably subdivided into several elementary sub-problems, being solved separately. This subdivision is raising up a question which mathematical theories are necessary for solving and right understanding the partial problems. Consequently, one has to find a direct reflection in formulating the parts of the curricula of the geometric course for students of mechanical engineering. And in addition, we have to select some didactic support using modern tools as dynamic-geometry-software and computer-algebra-systems (see Section 3).

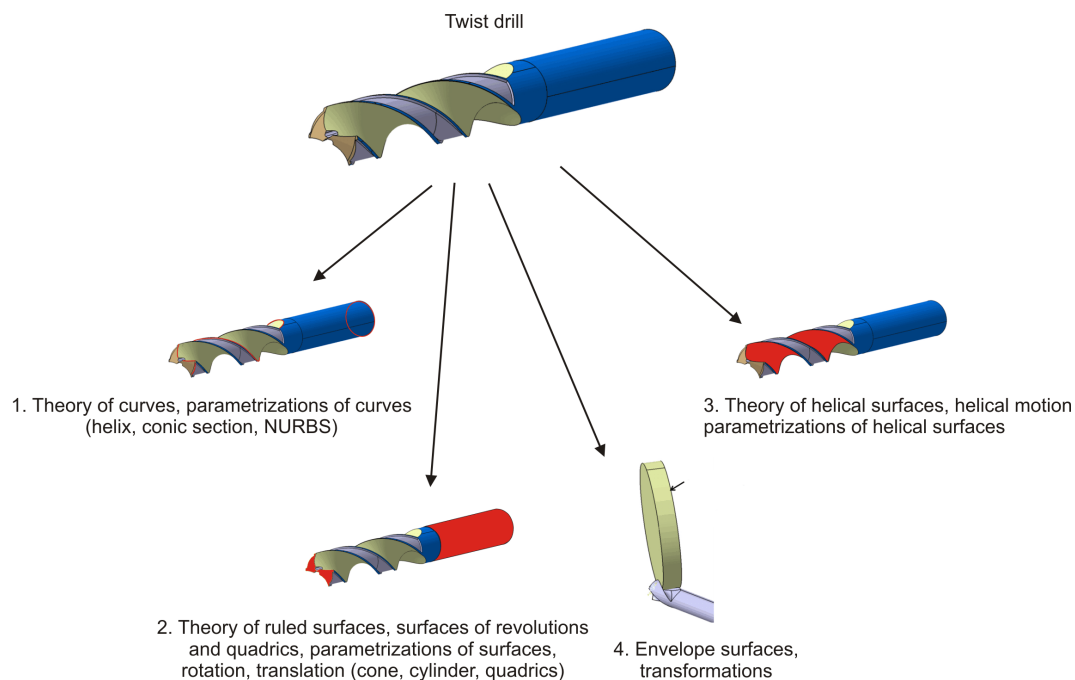


Figure 9: Diagram – subdivision of the real engineering problem into several elementary sub-problems and corresponding mathematical theories

Based on the observations summed up above, the course must be focused on geometry of curves and surfaces in close connection to mechanical engineering, namely to mechanics, processing, design,



robotics and CAD/CAM application – of course, along with all prerequisites as e.g. linear algebra and calculus. The main aim of the course is to show how modern geometric methods and algorithms contribute to studying and solving real problems of technical practice (see [3], [9]).

The course aims at giving the student a firm understanding of analytic method in Euclidean geometry, geometric transformations in the plane and in the space, fundamental theory of curves and surfaces including methods of their visualization using classical methods of descriptive geometry, as well as using 3D computer graphics software, and finally it enables the student to apply the methods in engineering and in design. After passing the course, student should be able to describe curves and surfaces which are used in mechanical engineering and he/she is able to construct orthogonal projection of this objects and to solve geometric problems on surfaces (tangent objects, curvatures, cuts, intersection curves etc.). The students are also used to apply fundamental methods of computational geometry, scientific computing and geometric modelling using suitable software tools.

For the sake of brevity, we have chosen only a few topics; especially to demonstrate the efficient support of the teaching process via DGS and CAS – see the following figures illustrating the geometry of curves (Fig. 10 and Fig. 11), geometry of surfaces (Fig. 12, Fig. 13 and Fig. 14) and geometric transformations (Fig. 15). Clearly it is seen from the pictures that students must understand the techniques of descriptive geometry (especially axonometry and Monge projection).

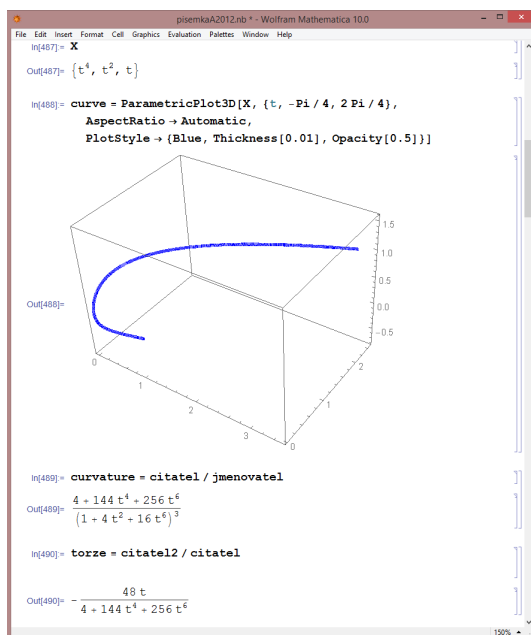


Figure 10: Computing the curvature of a curve (in Wolfram Mathematica)

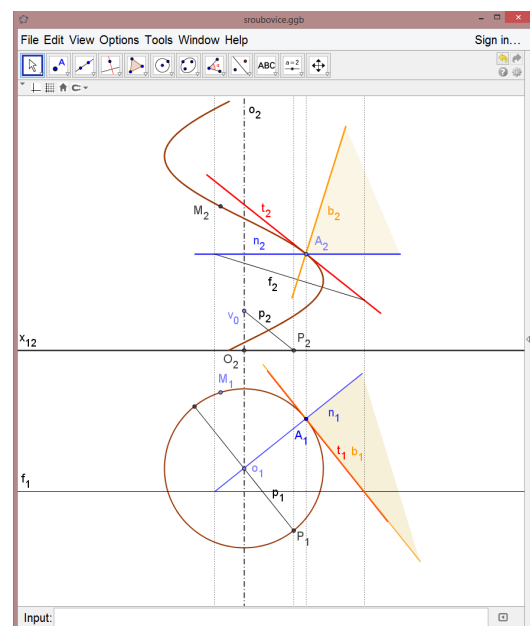


Figure 11: The helix and its Frenet frame (in GeoGebra)

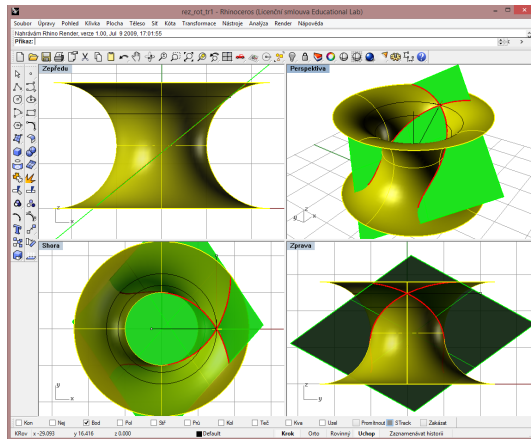


Figure 12: A surface of revolution and its tangent plane (in Rhinoceros)

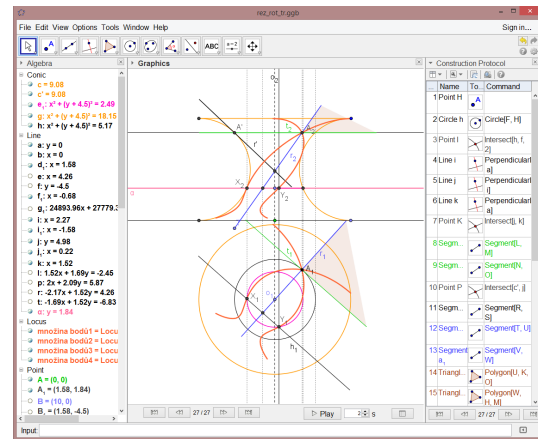


Figure 13: A surface of revolution and its tangent plane (in GeoGebra)

## 5 Conclusion

In this paper, we presented our experience with the problem-based geometry teaching at the Faculty of mechanical engineering. We discussed the question how dynamic geometric software GeoGebra along with the 2D modeler Rhinoceros and the computer algebra system Wolfram Mathematica can serve as a first step to the world of geometric modelling and computer-aided design. The approach was shown on one particular example taken from the course for mechanical engineering at the University of West Bohemia in Pilsen. First, we presented a real engineering problem, then we subdivided it to a number of elementary sub-problems and finally we demonstrated how the curriculum and the corresponding teaching procedures can be formulated to satisfy the goal to teach the students to solve the real technical problems. We also presented our experience with the problem-based teaching of geometry using dynamic geometry software, discussed some aspects of this approach and provided several particular examples/figures showing the approach.

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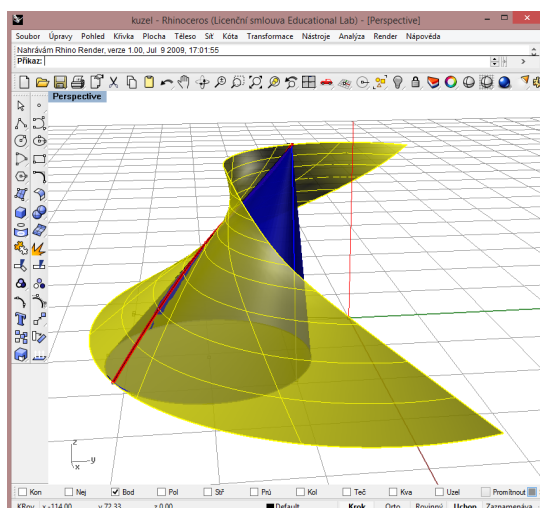


Figure 14: A envelope surface and its generatrix (in Rhinoceros)

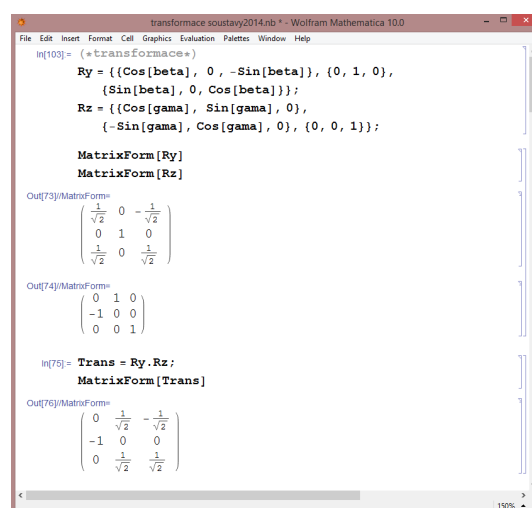


Figure 15: Computing with transformations (in Wolfram Mathematica)

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