

# Extremal Polynomials with Mathematica: An Elementary Approach

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# Background [CADGME14–Vajda]

Univ of Szeged, Bolyai Inst., Numeric and Symbolic Computation

Work supported by  
IPA Crossborder Project HUSRBI203/221/024&  
TAMOP Telemedicina (Univ. of Szeged)



## Outline [CADGME14–Vajda]

Definition: What is an extremal polynomial?

Classical case: Chebyshev Polynomials

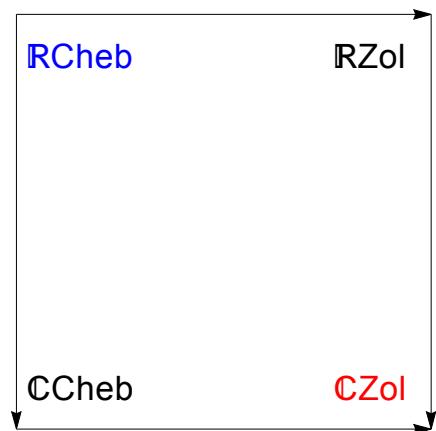
Possible generalizations: other sets, complex problems, Zolotarev problems, Markov inequalities...

Methods for solving the problems (symbolic computation)

Connections between the solutions

Applications

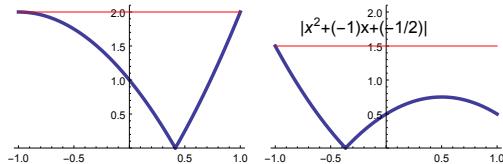
## Outline2 [CADGME14–Vajda]



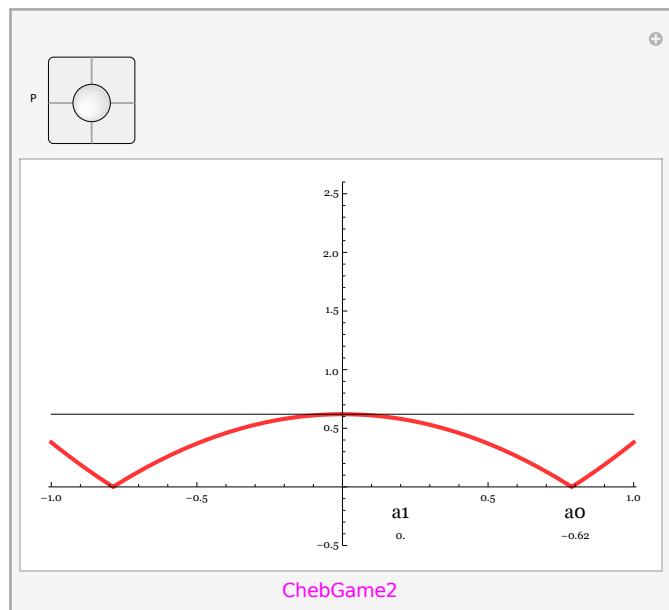
## Intro [CADGME14–Vajda]

**Definition:** The classical case

- Given: the interval  $I = [-1, 1]$ ; degree  $n$ , the set of monic polynomials of degree  $n$
- Find: the extremal (minimal) polynomial  $p^* = x^n + \dots + a_1 x + a_0$  which deviates least from the zero polynomial, i.e. the maximum of deviation on  $I$  is minimal; optimization problem, mimimax problem (difficult)



## Game [CADGME14 —Vajda]



ChebGame2

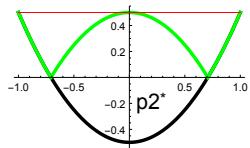
## Intro [CADGME14–Vajda]

The classical case, Solution: scaled Chebyshev polynomials

- Find: the extremal (minimal) polynomial  $p^* = x^n + \dots + a_1 x + a_0$  which deviates least from zero on  $I$

`Minimize[Maximize[{(x2 + a1 x + a0 - 0)2, -1 ≤ x ≤ 1}, x][[1]], {a0, a1}]`

$$\left\{ \frac{1}{4}, \left\{ a0 \rightarrow -\frac{1}{2}, a1 \rightarrow 0 \right\} \right\}$$

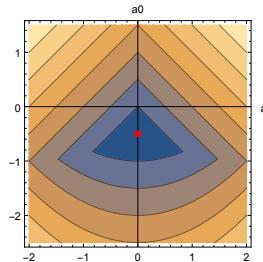


## Intro [CADGME14–Vajda]

The classical case, Solution: scaled Chebyshev polynomials

- Find: the extremal (minimal) polynomial  $p^* = x^n + \dots + a_1 x + a_0$  which deviates least from zero on  $I$ ,

$n = 2$ .



- $\max(|1 + a_1 + a_0|, |1 - a_1 + a_0|, |a_0 - a_1^2 / 4|)$

## Chebyshev Polys [CADGME14]

The classical case, Solution: Real Scaled Cheyshev Polynomials (1st type) – easily computable (rat. coeff./rec.)

- unique solution: scaled quadratic Cheyshev polynomial

$$x^2 + a1 x + a0 /. \{a0 \rightarrow -1/2, a1 \rightarrow 0\}$$

$$-\frac{1}{2} + x^2$$

$$1 \quad x$$

$$2 \quad -\frac{1}{2} + x^2$$

$$3 \quad -\frac{3x}{4} + x^3$$

$$4 \quad \frac{1}{8} - x^2 + x^4$$

## Approach 1 [CADGME14–Vajda]

Approximation by discretization (making the problem finite)

- Given: the interval  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 6$ , the set of monic polynomials of degree  $n$
- Find: the extremal (minimal) polynomial  $p^* = x^6 + a_5 x^5 \dots + a_1 x + a_0$  which deviates least from zero

```
SPL1 = Flatten[{Table[{-n, n}, {n, 1/2, 1, 1/4096}]}];
p6[x_] = Sum[a[j] x^j, {j, 0, 5}] + 32 x^6
32 x^6 + a[0] + x a[1] + x^2 a[2] + x^3 a[3] + x^4 a[4] + x^5 a[5]
CL1 = Map[-m <= p6[#] <= m &, SPL1];
```

# Approach 1 [CADGME14–Vajda]

Approximation by discretization (making the problem finite)

```
NMinimize[{m, m > 0, CL1}, {m, a[0], a[1], a[2], a[3], a[4], a[5]}]
{0.421875, {m → 0.421875, a[0] → -5.70313, a[1] → 7.762×10-11, a[2] → 34.125,
a[3] → -1.84323×10-10, a[4] → -60., a[5] → 1.06706×10-10}}

Round[{m, a[4], a[2], a[0]}] /.
NMinimize[{m, m > 0, CL1} /. {a[1] → 0, a[3] → 0, a[5] → 0}, {m, a[0], a[2], a[4]}][[2]], 10^(-6)]
{27/64, -60, 273/8, -365/64}

% / 2^5
{27/2048, -15/8, 273/256, -365/2048}
```

## Approach 2 [CADGME14–Vajda]

Symbolic Computation: Quantifier Elimination (QE)

- Given: the interval  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 6$ , Find the extremal poly  $p^*$

```
pr1 = Resolve[0 < m < 1 & ForAll[x, 1/4 < x < 1, x^3 + a2 x^2 + a1 x + a0 ≤ m], Reals];  
pr2 = Resolve[0 < m < 1 & ForAll[x, 1/4 < x < 1, -m <= x^3 + a2 x^2 + a1 x + a0], Reals];
```

## Approach 2 [CADGME14–Vajda]

Symbolic Computation: Quantifier Elimination (QE)

- Given: the interval  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 6$ , Find the extremal poly  $p^*$  (C. Brown)

```
CylindricalDecomposition[Resolve[Exists[{a0, a1, a2}, pr1 \[And] pr2], Reals], m]
```

$$\frac{27}{2048} \leq m < 1$$

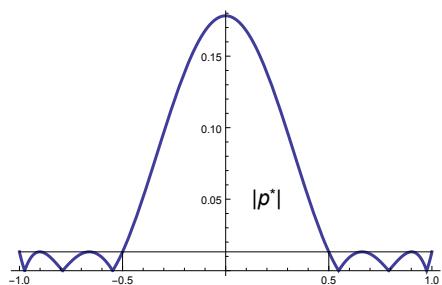
```
CylindricalDecomposition[pr1 \[And] pr2 /. m -> 27 / 2048, {a2, a1, a0}]
```

$$a_2 = -\frac{15}{8} \quad \& \quad a_1 = \frac{273}{256} \quad \& \quad a_0 = -\frac{365}{2048}$$

## Approach 2 [CADGME14–Vajda]

Symbolic Computation: Quantifier Elimination (QE)

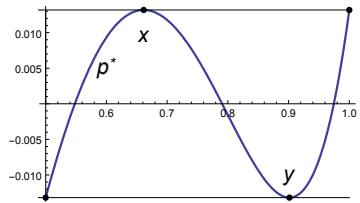
- Given: the interval  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 6$ , Find the extremal poly  $p^*$



## Approach 3 [CADGME14–Vajda]

Symbolic Computation: Groebner Basis (GB)

Theorem: Equiocillation property of the optimal polynomial



$$p6[x_] = x^6 + a4 x^4 + a2 x^2 + a0;$$

```
polysys = {p6[1] + p6[1/2], Factor[D[p6[x], x]][[-1]], Factor[p6[x] - p6[1]][[-1]],
p6[y] + p6[1], Factor[D[p6[x], x] /. x -> y][[-1]], p6[1] - m};
```

## Approach 3 [CADGME14–Vajda]

Symbolic Computation: Groebner Basis (GB)

```
Factor[First[GroebnerBasis[polysys, {x, y, a2, a4, a0, m}]]]
m (-27 + 32 m)2 (27 + 256 m)2 (-27 + 2048 m)

Factor[First[GroebnerBasis[polysys /. m → 27 / 2048, {x, y, a2, a4, a0}]]]
365 + 2048 a0
```

## Generalization [CADGME14]

Two symm. intervals: Even Degree easy, Odd Degree difficult

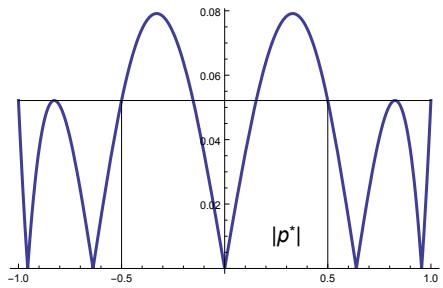
- Given:  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 5$ , the set of monic polynomials of degree (at most)  $n$ , Find the extremal poly  $p^*$  Norms and coefficients are not rationals!  $\rightarrow$

```
Resolve[0 < m < 1 \[And] Exists[{b1, b3},
  ForAll[x, 1/2 < x < 1, -m <= x^5 + b3 x^3 + b1 x <= m], Reals]
Root[27 + 432 #1 - 18 432 #1^2 + 4096 #1^3 &, 2] \leq m < 1
Root[27 + 432 #1 - 18 432 #1^2 + 4096 #1^3 &, 2] // N
0.0521374
```

## Generalization [CADGME14]

### Symbolic Computation

- Given:  $E = [-1, -1/2] \cup [1/2, 1]$ ;  $n = 5$ , the set of monic polynomials of degree (at most)  $n$ , Find the extremal poly  $p^*$



## Generalization [CADGME14]

### Symbolic Computation

- Given: a circular arc  $A^{2\pi/3} = \{z \mid |z|=1 \quad |\arg z| \leq 2\pi/3\}$ ;  $n = 4$ , the set of monic polynomials of degree (at most)  $n$
- Find: the extremal (minimal) polynomial  $p^* = z^4 + a_4 z^3 \dots + a_1 x + a_0$  ( $a_j \in \mathbb{C}$  !) which deviates least from the zero polynomial, i.e., the maximum of deviation on  $I$  is minimal

## Generalization [CADGME14]

### Symbolic Computation

- 1) coeffs are reals (symm) 2)  $|p(z)|$  can be expressed on the arc  $A^{2\pi/3}$  by  $x = \operatorname{Re}(z)$  3)  $|p|$  is max at the two endpoints, without going into details...

```
e4 = Plus @@ Map[##^2 &,
  CoefficientList[ComplexExpand[a0 + a1 z + a2 z^2 + a3 z^3 + z^4 /. z → x + y I] /. I → z, z]]
  (a1 y + 2 a2 x y + 3 a3 x^2 y + 4 x^3 y - a3 y^3 - 4 x y^3)^2 +
  (a0 + a1 x + a2 x^2 + a3 x^3 + x^4 - a2 y^2 - 3 a3 x y^2 - 6 x^2 y^2 + y^4)^2
e4b = e4 /. y → Sqrt[1 - x^2] // Expand
1 + 2 a0 + a0^2 + a1^2 - 2 a2 - 2 a0 a2 + a2^2 - 2 a1 a3 + a3^2 - 6 a1 x + 2 a0 a1 x + 2 a1 a2 x + 2 a3 x - 6 a0 a3 x +
2 a2 a3 x - 16 a0 x^2 + 4 a2 x^2 + 4 a0 a2 x^2 + 4 a1 a3 x^2 + 8 a1 x^3 + 8 a0 a3 x^3 + 16 a0 x^4
```

# Generalization [CADGME14]

## Symbolic Computation

```

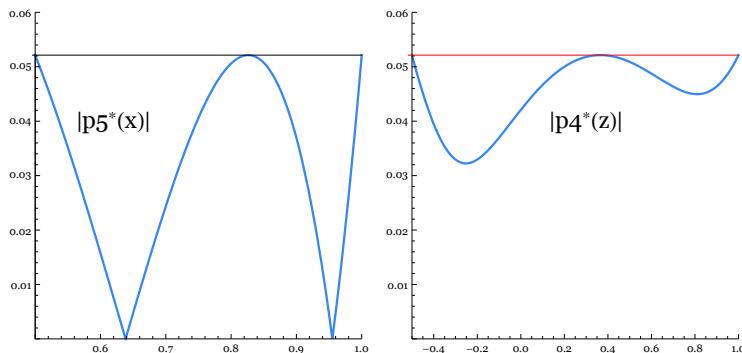
Resolve[834/1000 < m < 835/1000 \[And] Exists[{a2, a3}, ForAll[x, -1/2 <= x <= 1, e4b <= m^2]] /. a0 \[Rule] m - 1 - a1 - a2 - a3 /.
a1 \[Rule] 1/2 (-2 - a2 + m + Sqrt[-3 a2^2 + 2 a2 m + m^2]), Reals]
Root[27 + 27 #1 - 72 #1^2 + #1^3 &, 2] \leq m < 167/200
{snorm = RootReduce[Root[27 + 27 #1 - 72 #1^2 + #1^3 &, 2]/2^4], N[snorm]}
{Root[27 + 432 #1 - 18432 #1^2 + 4096 #1^3 &, 2], 0.0521374}

```

the norm is the same as the norm for the two real intervals, deg5! →

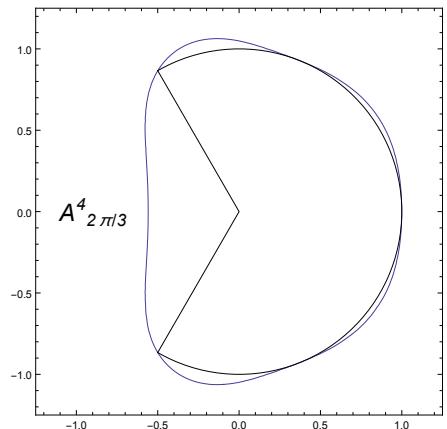
## Generalization [CADGME14]

Comparison of the real and complex results



## Generalization [CADGME14]

Summary: complex solution as contour line



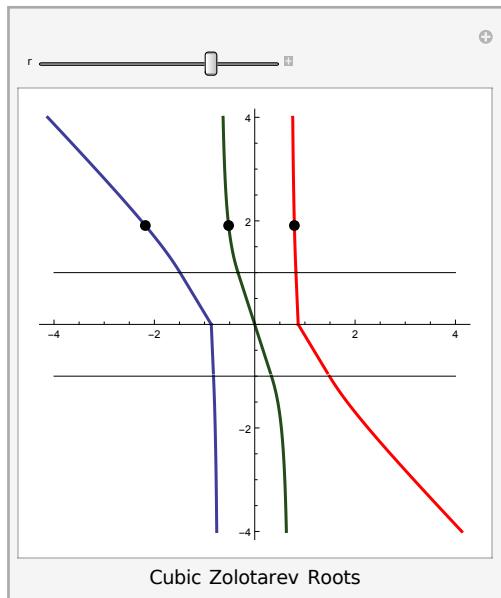
## Generalization [CADGME14]

(Real) Zolotarev Polynomials

- Problem  
[-1,1],  $r \in \mathbb{R}$ ,  $x^3 + r x^2 - a_1 x - a_0$   
Goal: Find  $a_0, a_1$  such that the monic cubic polynomial has the least deviation from the zero polynomial.
- Example: Root Orbits

## Generalization [CADGME14]

(Real) Zolotarev Polynomials, Root Orbits



## Generalization [CADGME14]

(Real) Zolotarev Polynomials, Root Expressions

$$\begin{aligned}
 R3[r_] := \text{Piecewise}\left[ \left\{ \left\{ \left\{ \left\{ \text{Root}\left[ -18r - r^3 - \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 1 \right], r \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \text{Root}\left[ -18r - r^3 - \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 2 \right], r \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \text{Root}\left[ -18r - r^3 - \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 3 \right], r \right\}, r \leq -1 \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \left\{ \frac{1}{6} \left( -2r - \sqrt{3} \sqrt{9 - 6r + r^2} \right), r \right\}, \{-r/3, r\}, \left\{ \frac{1}{6} \left( -2r + \sqrt{3} \sqrt{9 - 6r + r^2} \right), r \right\} \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. -1 < r < 0 \right\}, \left\{ \left\{ \frac{1}{6} \left( -2r - \sqrt{3} \sqrt{9 + 6r + r^2} \right), r \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ -r/3, r \right\}, \left\{ \frac{1}{6} \left( -2r + \sqrt{3} \sqrt{9 + 6r + r^2} \right), r \right\} \right\}, 0 \leq r < 1 \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \left\{ \text{Root}\left[ -18r - r^3 + \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 1 \right], r \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \text{Root}\left[ -18r - r^3 + \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 2 \right], r \right\}, \right. \right. \right. \right. \\
 \left. \left. \left. \left. \left\{ \text{Root}\left[ -18r - r^3 + \sqrt{(3+r^2)^3 - 27\#1 + 27r\#1^2 + 27\#1^3} \&, 3 \right], r \right\}, r \geq 1 \right\} \right\} \right] \right]
 \end{aligned}$$

## Summary [CADGME14 —Vajda]

Extremal Polynomials as Field for Explorations with computer (course material, possible research topic, see web)

Symbolic, numeric and hybrid approaches for solution

A place for visualization and conjecturing

A variety of problems with good computational scalability

Elliptic functions and advanced hybrid techniques on a more advanced level