Visual introduction to bifurcations

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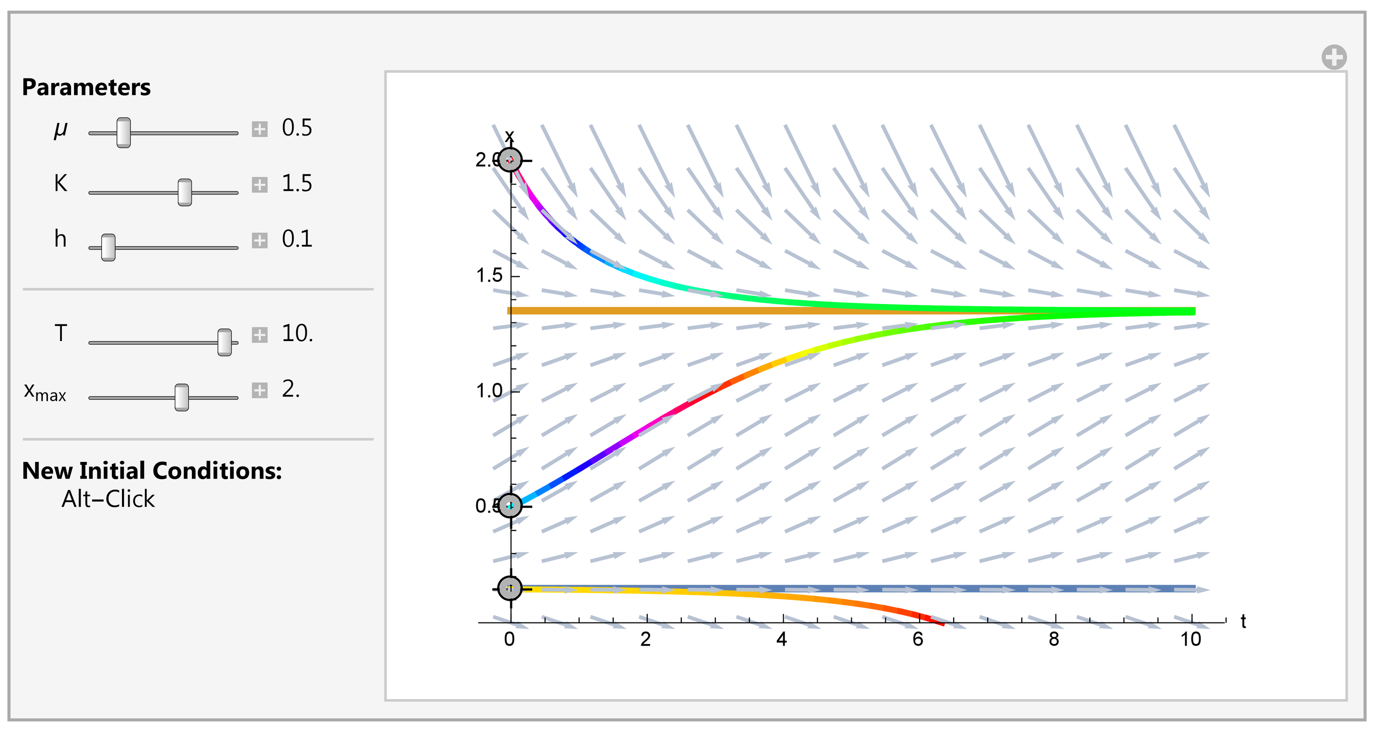
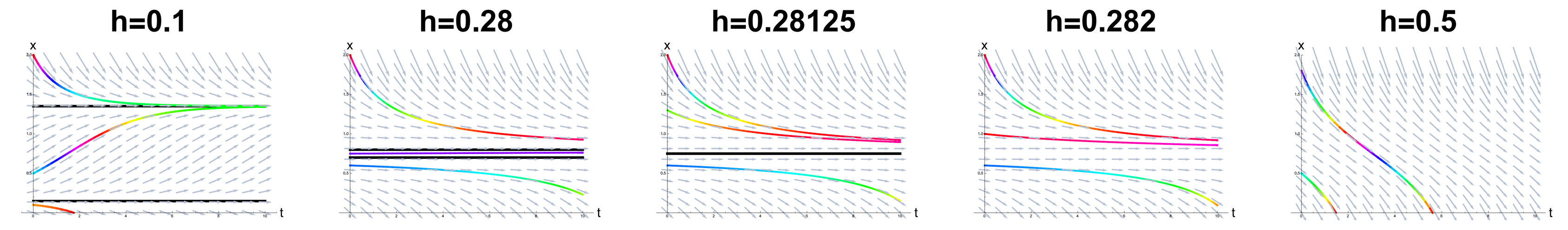
Introduction: a harvesting model in population dynamics

Model: $x'(t) = \mu x(t)\,(K - x(t)) - h$, where μ is the growth rate, K is the carrying capacity, h is the harvesting rate.



Fix the parameters $\mu = 0.5$ and $K = 1.5$, then some snapshots:



Observe: the structure of the solutions may be different with changing of parameter!

Concepts of bifurcation theory

Definition: A bifurcation appears at α_0 in a parameter-dependent system, if the dynamics of the system changes qualitatively when the parameter is passing through α_0.
Remark: This phenomenon appears in many real life problems!
Goal: find simple models (=normal forms) for bifurcations, which can be studied with elementary tools and represents the family of models, where the bifurcation can occur.

Difficulties in education

- Models are quite complicated; problem with finding of the simple models;
- finding of the parameter, which causes structural changes;
- mathematical study of more-parameter bifurcation is a hard challenge;
- deep mathematical theories are required, even in the simplest cases.

Didactic main points

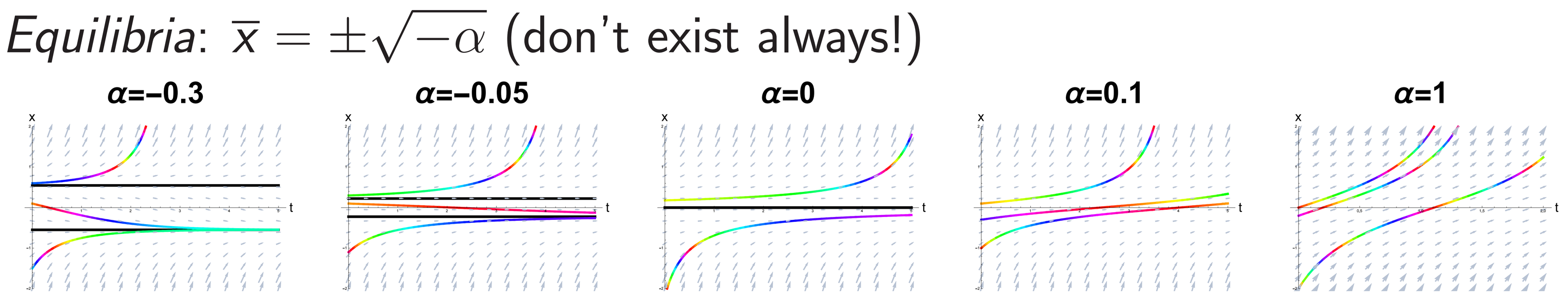
- Mathematical knowledge vs. necessity: students should investigate this phenomenon, but don't know the necessary theory;
- dealing with simple models representing bifurcations;
- avoiding deep theories;
- dynamic visualization.

Advantages of Wolfram Mathematica

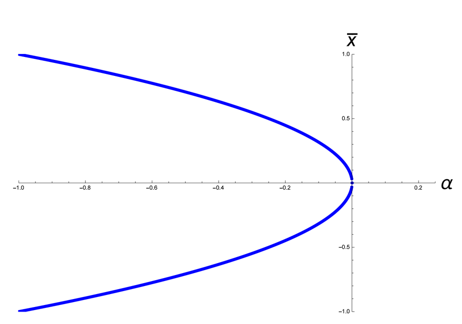
- Advanced visualization tools (StreamPlot, ParametricPlot,...);
- advanced tools to study parameter-dependent systems: ParametricNDSolve;
- dynamic modules help exploring the bifurcations (Manipulate).

Saddle-node bifurcation in system $x'(t) = \alpha + x^2(t)$

- *Equilibria*: $\overline{x} = \pm\sqrt{-\alpha}$ (don't exist always!)



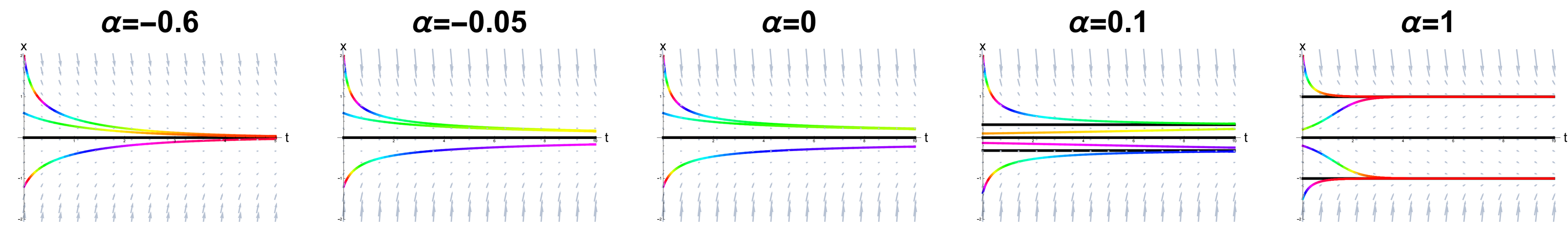
- *Bifurcation diagram*:



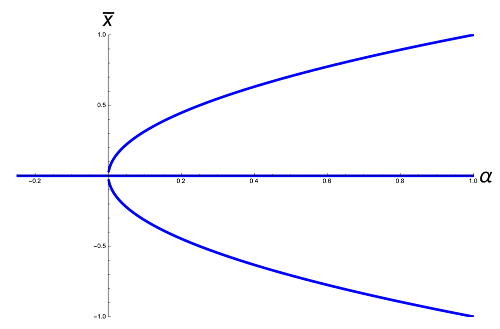
- *Remark*: in the previous harvesting model a saddle-node bifurcation appears at $h = \frac{K^2\mu}{4}$ $\Rightarrow$ the system can be transformed to the form $x' = \alpha - x^2$.
- *Interpretation*: disappearance of stable equilibrium = **sudden** extinction of the population $\Rightarrow$ only controlled harvesting is allowed to do!

Pitchfork bifurcation in system $x'(t) = \alpha x - x^3(t)$

- *Equilibria*: $\overline{x} = 0$ (always exists), $\overline{x} = \pm\sqrt{\alpha}$



- *Bifurcation diagram*:



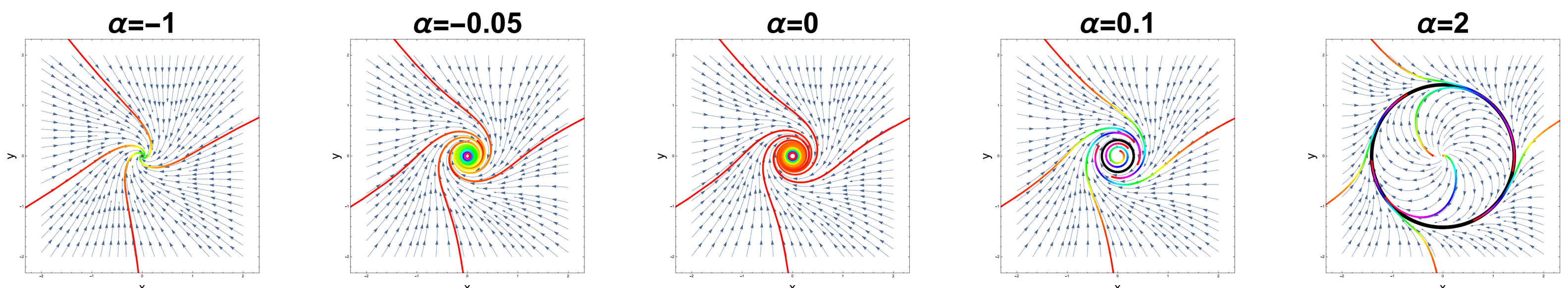
- Bistability occurs in biological systems = hysteretic systems.
- *Remark*: for saddle-node bifurcation there are unbounded solutions vs. for pitchfork bifurcation all solution converges $\Rightarrow$ at the computer-aided study of saddle-node bifurcation we should be more careful!

Bifurcation in a two-dimensional system: Hopf-bifurcation

- **Normal form**:

$$x'(t) = \alpha x(t) + y(t) - x(t)\left(x^2(t) + y^2(t)\right)$$
$$y'(t) = -x(t) + \alpha y(t) - y(t)\left(x^2(t) + y^2(t)\right)$$

- *New object*: periodic solution! Some snapshots in phase space:

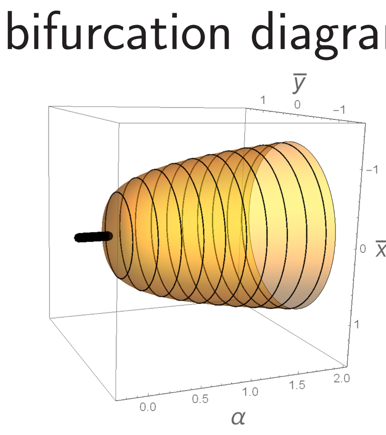


- The system after a polar transformation:

$$r'(t) = r(t)(\alpha - r^2(t))$$
$$\varphi'(t) = 1$$

- Closed curves (=periodic solutions) in phase space: circles with radius $\sqrt{\alpha}$ and centre $(0, 0)$.

- A bifurcation diagram:



- Important phenomenon vs. low-level of knowledge about higher dimensional mathematics (differential geometry, multivariable analysis).
- *Remark for engineers*: in the industry it may be catastrophic, if the behavior of a machine produces a Hopf-like transition.

References

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